Multifractional, multistable, and other processes with prescribed local form

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Abstract

We present a general method for constructing stochastic processes with prescribed local form. Such processes include variable amplitude multifractional Brownian motion, multifractional α-stable processes, and multistable processes, that is processes that are locally $\alpha(t)$-stable but where the stability index $\alpha(t)$ varies with t. In particular we construct multifractional multistable processes, where both the local self-similarity and stability indices vary.

1 Introduction

In this paper we present a general framework for constructing stochastic processes with prescribed local forms.

Stochastic processes where the local Hölder regularity varies with a parameter t (usually time) are important both in theory and in practical applications. The best known example is multifractional Brownian motion (mBm), where the Hurst index h of fractional Brownian motion is replaced by a functional parameter $h(t)$, permitting the Hölder exponent to vary in a prescribed manner. This allows local regularity and long range dependence to be decoupled to give sample paths that are both highly irregular and highly correlated, a useful feature for terrain or TCP traffic modeling.

For modelling financial or medical data another feature is often important, namely the presence of jumps. Stable non-Gaussian processes give good models for data containing discontinuities, with the stability index α controlling the distribution of jumps. Recently, multifractional stable processes, generalising mBm, were introduced to provide jump processes with varying local regularity. However, a further step is needed for situations where both local regularity and jump intensity vary with time, for example to model financial data or epileptic episodes in EEG, where for some periods there may be only small jumps and at other instants very large ones. Our method may be used to construct processes where both h and α vary in a prescribed way: thus there are two parameters which might correspond to distinct aspects of financial risk, to different sources of irregularity leading to the onset of epilepsy, or to textured images where both Hölder regularity and the distribution of discontinuities varies.

It is natural to construct processes $Y = \{Y(t) : t \in \mathbb{R}\}$ that have an identifiable local form near each u, that is where there is a limiting process

$$\lim_{r\to 0} \frac{Y(u+rt) - Y(u)}{r^h} = Y'_u(t) \tag{1.1}$$

which may vary with u. If this limit exists as a non-trivial process we will say that Y is *h-localisable* at u and call the process $Y'_u = \{Y'_u(t) : t \in \mathbb{R}\}$ the *local form* of Y at u. The limit (1.1) may be taken in several ways: of particular interest are convergence in finite dimensional distributions, and convergence in distribtion; in the latter case we term the process *strongly h-localisable*. We will be especially concerned with h-localisable processes with $0 < h < 1$ which are usually of a fractal nature.

The most familiar example is multifractional Brownian motion Y which resembles index-$h(u)$ fractional Brownian motion close to time u but where $h(u)$ varies, that is

$$\lim_{r\to 0} \frac{Y(u+rt) - Y(u)}{r^h} = B_{h(u)}(t) \tag{1.2}$$

where B_h is index-h fractional Brownian motion, see [1, 2, 3, 10, 14]. Generalising this, mulitfractional α-stable processes processes have been constructed with local form $h(u)$-self-similar linear α-stable motions [19, 20].

It is clear from (1.1) that the h-local form Y'_u at u, if it exists, must itself be h-self-similar, that is $Y'_u(rt) = r^h Y'_u(t)$ for $r > 0$. However, much more is true: under quite general conditions Y'_u must be self-similar with stationary increments (sssi) at almost all u at which it is strongly localisable, that is $r^{-h}(Y'_u(u+rt) - Y'_u(u)) = Y'_u(t)$ for all u and $r > 0$, see [8, 9]. Thus if we wish to construct processes with given local forms, the local forms should themselves be sssi. Whilst this is a strong requirement, many classes of sssi processes are known, including fractional Brownian motion, linear fractional stable motion and α-stable Lévy motion, see [6, 17].

Our general construction will allow known localisable processes $X(\cdot, v) = \{X(t, v) : t \in \mathbb{R}\}$ for a range of v to be pieced together to yield a localisable 'diagonal' process $Y = \{X(t,t) : t \in \mathbb{R}\}$ with local form depending on t. We will obtain conditions for the transference of the local properties of $X(\cdot, v)$ to Y. The basic setting is akin to that adopted in [2, 19]. Thus we seek a random field $\{X(t, v) : (t, v) \in \mathbb{R}^2\}$ such that for each v the local form $X'_v(\cdot, v)$ of $X(\cdot, v)$ at v is the desired local form Y'_v of Y at v. Typically, for each v the process $\{X(t, v) : t \in \mathbb{R}\}$ will be one where the local form can be readily identified, such as an sssi process. Clearly the interplay of $X(\cdot, v)$ for v in a neighbourhood of u will be crucial to the local behaviour of Y near u. Thus the random field is set up as an integral or sum of functions that depend on t and v with respect to a single underlying random measure or process to provide the necessary correlations. In Section 4 we derive general criteria that guarantee the transference of localisability from the $X(\cdot, v)$ to $Y = \{X(t,t) : t \in \mathbb{R}\}$; Section 5 addresses this for strong localisability.

We illustrate the general method with several specific classes of processes. The method permits easy constructions of *multifractional* processes such as multifractional Brownian motion with variable amplitude (Section 6) and multifractional α-stable motions (Section 7). In Section 9 we develop *multistable* processes, where the stability index $\alpha(t)$ is allowed to vary. Here the constructions are based on sums over Poisson processes for which the required properties are reviewed in Section 8. In particular we construct *multifractional multistable* processes, where both the local self-similarity index and the stability index vary.

2 Convergence of random processes

We work with two definitions of localisability of real valued random processes, one in terms of convergence of finite dimensional distributions and one requiring the stronger convergence in distribution, appropriate when the sample functions are viewed as members of some metric space.

Given a probability space $(\Omega, \mathcal{P}, \mathsf{P})$, a *random process* X on a domain T is a family of random variables $\{X(t) : t \in T\}$. The law of the process is determined by its finite dimensional distributions, that is the k-dimensional distributions of $(X(t_1), \dots, X(t_k))$ for all $t_1, \dots, t_k \in T$ for all k. For our purposes T will be either $\mathbb{R}$ or a subinterval of $\mathbb{R}$, or sometimes a subset of $\mathbb{R}^2$ in which case we will refer to the process as a *random field.*

Let X_r be a family of random processes on T. We say that X_r converges to a process X in *finite-dimensional distributions*, written $X_r \stackrel{\text{fdd}}{\to} X$, if, for all k and all $t_1, \dots, t_k \in T$,

$$(X_r(t_1), \dots, X_r(t_k)) \to (X(t_1), \dots, X(t_k)) \tag{2.1}$$

as $r \to 0$ (or some other value) in the sense of k-dimensional distributions.

For processes with sample paths in suitable function spaces, convergence in distribution is defined in terms of a metric on the spaces. Let $C(T)$ be the space of continuous functions on $T \subset \mathbb{R}$. For T compact, let d^T be the uniform metric on $C(T)$, that is

$$d^T(x, y) = \sup_{t \in T} |x(t) - y(t)| \quad (x, y \in C(T)).$$

Then

$$d(x, y) = \sum_{\tau=1}^{\infty} 2^{-\tau} \min\{1, d^{[-\tau,\tau]}(x, y)\} \quad (x, y \in C(\mathbb{R})) \tag{2.2}$$

defines a seperable metric on $C(\mathbb{R})$ that gives the topology of uniform convergence on compact subsets of $\mathbb{R}$.

To accommodate processes with sample functions that have jumps, let T be a closed subinterval of $\mathbb{R}$, and let $D(T)$ denote the "càdlàg" functions on T, that is functions which are continuous on the right and have left limits at all $t \in T$. When T is a bounded closed interval we define a metric $d_S^{[a,b]}$ on $D[a, b]$ as follows. Let Φ be the class of strictly increasing continuous bijections from $[a, b]$ to itself. For each $x, y \in D[a, b]$ we define $d_S^{[a,b]}(x, y)$ to be the infimum of those $\delta > 0$ for which there exists $\phi \in \Phi$ such that both $\sup_{0 \le t \le 1} |\phi(t) - t| \le \delta$ and $\sup_{0 \le t \le 1} |x(t) - y(\phi(t))| \le \delta$. Then $d_S^{[a,b]}$ is the *Skorohod metric* on $D([a, b])$, see [15, Chapter VI] or [4]. The Skorohod metric extends to a seperable metric on $D(\mathbb{R})$ by

$$d_S(x, y) = \sum_{\tau=1}^{\infty} 2^{-\tau} \min\{1, d_S^{[-\tau,\tau]}(x, y)\} \quad (x, y \in D(\mathbb{R})). \tag{2.3}$$

Taking T as $[a, b]$ or $\mathbb{R}$, let $F(T)$ be either $C(T)$ or $D(T)$ with the appropriate metric as above. Given a probability space $(\Omega, \mathcal{P}, \mathsf{P})$ we call $X : \Omega \to F(T)$ a *random function* or *random element* of $F(T)$ if $X^{-1}(B) \in \mathcal{P}$ for every Borel subset B of the metric space $F(T)$. If T' is a suitable subset of T and X is a random function on T then we may regard the restriction of X as a random function on T'.

We will say that two random functions X, Y are equal and write $X = Y$ if either they are equal in finite dimensional distributions or if they are equal in distribution; it will generally be clear from the context which is intended.

Convergence in distribution in these spaces is stronger than convergence of finite dimensional distributions, with more global control on the approach to the limit. For X_r and X random functions in $F(T)$ where T is a closed interval, perhaps $\mathbb{R}$, we say that X_r *converges in distribution* to X, written $X_r \xrightarrow{\mathrm{d}} X$, if $\mathsf{E}(f(X_r)) \to \mathsf{E}(f(X))$ for all bounded continuous $f : F(T) \to \mathbb{R}$. Convergence in distribution is equivalent to convergence of finite dimensional distributions together with an appropriate stochastic equicontinuity condition. For example, in the case of $C(T)$ where T is a compact interval this additional condition may be expressed: given $c > 0$ there exists $\delta > 0$ such that

$$\lim_r \mathsf{P}\left(\sup_{t,t' \in T, |t-t'|<\delta} |X_r(t) - X_r(t')| > c \right) = 0. \tag{2.4}$$

There is a similar characterisation for $D(T)$, see [4, 15].

Note that whilst convergence in distribution in $C(\mathbb{R})$ or $D(\mathbb{R})$ is defined as convergence in the metric d or d_S, this is equivalent to convergence in distribution of the restrictions of the random functions to $[a, b]$ for all compact intervals $[a, b]$.

A technicality here is that our functions or processes may have a domain U that is a proper interval of $\mathbb{R}$. However X_r will generally be a sequence of enlargements of a process about some u interior to U, for example $X_r(t) = (Y(u + rt) - Y(u))/r^h$ as $r \to 0$. In such cases, as we approach the limit, the domain of definition will eventually include every finite subset $\{t_1, \ldots, t_k\}$ of $\mathbb{R}$ so we may still refer to convergence of finite dimensional distributions to a process on $\mathbb{R}$. Similarly, the domain will eventually contain each bounded interval $[a, b]$, allowing us to identify convergence in distribution of such sequences as convergence in distribution of the restrictions of the random functions to all bounded intervals.

Finally, recall that *convergence in probability* of random functions, written $X_r \xrightarrow{\mathrm{p}} X$, requires that for all $c > 0$

$$\lim_r \mathsf{P}\left(d_0(X_r, X) > c\right) \to 0$$

for $X_r, X \in F(T)$, where d_0 is the appropriate metric.

3 Localisable processes

For convenience we give the definitions of localisability at u for random processes with domain $\mathbb{R}$, but the definitions will also apply in the obvious way where the domain is a real interval with u as an interior point. Intuitively, a random process Y on $\mathbb{R}$ is localisable at $u \in \mathbb{R}$ if it has a unique non-trivial scaling limit at u. More precisely, we say that $Y = \{Y(t) : t \in \mathbb{R}\}$ is *h-localisable* at u with *local form* the random process $Y'_u = \{Y'_u(t) : t \in \mathbb{R}\}$, if

$$\frac{Y(u+rt) - Y(u)}{r^h} \to Y'_u(t) \tag{3.1}$$

as $r \searrow 0$, where convergence is of finite dimensional distributions. If Y and Y'_u have versions in $C(\mathbb{R})$ or $D(\mathbb{R})$ and convergence in (3.1) is in distribution, we say that Y is *strongly localisable* at u with *strong local form* Y'_u. Of course, strongly localisable processes are localisable, with the strong local form a version of the local form in $C(\mathbb{R})$ or $D(\mathbb{R})$. Note that the term *locally asymptotically self-similar* is sometimes used for strong localisability.

A number of well-known processes are h-localisable, in particular processes that are *h-self-similar*, that is $Y(rt) = r^h Y(t)$ for $r > 0$, and which have *stationary increments*, that is $Y(t+u) = Y(t)$ for $u \in \mathbb{R}$.

Proposition 3.1 *Let $\{Y(t) : t \in \mathbb{R}\}$ be a process that is h-self-similar with stationary increments (h-sssi). Then Y is h-localisable at all $u \in \mathbb{R}$ with $Y'_u = Y$. If in addition Y is in $C(\mathbb{R})$ or $D(\mathbb{R})$ then Y is strongly h-localisable at all $u \in \mathbb{R}$.*

Proof. If Y is h-self-similar with stationary increments, then

$$\frac{Y(u+rt) - Y(u)}{r^h} = \frac{Y(rt) - Y(0)}{r^h} = \frac{Y(rt)}{r^h} = Y(t),$$

for all $r \neq 0$, so Y is localisible at u.

Further, if Y is in $C(\mathbb{R})$ or $D(\mathbb{R})$ then $(Y(u+rt) - Y(u))/r^h$ and $Y(t)$ have identical probability distributions, since probability distributions on $C(\mathbb{R})$ and $D(\mathbb{R})$ are completely determined by their finite dimensional distributions, see [4]. Thus Y is strongly localisible. ■

There are several important processes which are sssi so which are strongly localisable by Proposition 3.1.

For $0 < h < 1$, index-h fractional Brownian motion (fBm) on $\mathbb{R}$ may be defined as a stochastic integral with respect to Wiener measure W:

$$B_h(t) = c(h)^{-1} \int_{-\infty}^{\infty} \left((t-x)_+^{h-1/2} - (-x)_+^{h-1/2} \right) W(dx), \tag{3.2}$$

where $(a)_+ = \max\{0, a\}$ and $c(h)$ is a normalising constant that ensures that the variance $\mathrm{var} B_h(1) = 1$. (Here, and throughout, we make the convention that expressions involving the difference of two positive parts represent an indicator function when the exponent is 0, so for example, if $h = 1/2$ then $(t-x)_+^{h-1/2} - (-x)_+^{h-1/2}$ is taken to mean $\mathbf{1}_{[0,t)}(x)$.) It is well-known [6, 7, 13, 17] that index-h fBm is an h-self-similar process with a version in $C(\mathbb{R})$ that has stationary increments, so is strongly localisable at all $u \in \mathbb{R}$ with $(B_h)'_u = B_h$.

The α-stable processes form another important class of fractal processes of $C(\mathbb{R})$, or of $D(\mathbb{R})$ in the case of 'jump' processes, see Section 7. Under certain conditions α-stable processes may be sssi, see [17, Corollary 7.3.4], in which case by Proposition 3.1 they are strongly h-localisable.

A particular instance is linear stable fractional motion:

$$L_{\alpha,h}(t) = \int_{-\infty}^{\infty} \left[a \left((t-x)_+^{h-1/\alpha} - (-x)_+^{h-1/\alpha} \right) + b \left((t-x)_-^{h-1/\alpha} - (-x)_-^{h-1/\alpha} \right) \right] M(dx), \tag{3.3}$$

where $0 < \alpha < 2$ and M is an α-stable random measure with constant skewness β and control measure Lebesgue measure, $0 < h < 1$ and a and b are constants, see [17, Section 7.4 and Chapter 10]. The process is h-sssi and so is h-localisable at all $u \in \mathbb{R}$ with $(L_{\alpha,h})'_u = L_{\alpha,h}$. Provided that $h > 1/\alpha$ it has a version in $C(\mathbb{R})$, so is strongly localisable. However, if $h < 1/\alpha$ then almost surely Y is unbounded on every interval and so is not a process of $D(\mathbb{R})$, though it is nevertheless localisable. (Note that later we will represent such processes as Poisson sums rather than integrals with respect to random measures.)

An α-stable Lévy motion, $0 < \alpha < 2$ is a process in $D(\mathbb{R})$ with stationary independent increments which have a strictly α-stable distribution. It may be represented as

$$L_\alpha(t) = M([0,t]) \tag{3.4}$$

where M is an α-stable random measure on $\mathbb{R}$ with constant skewness intensity, see [17, Section 7.5]. Then L_α is $1/\alpha$-sssi, and so is strongly $1/\alpha$-localisable.

In later sections we will give general constructions of localisable processes where the local form Y'_u varies with u. For now we note that localisability behaves well under reasonably smooth changes of coordinates. In particular the following proposition allows the introduction of varying 'local amplitude' for localisable processes.

Proposition 3.2 *Let U be an interval with u an interior point. Suppose that $\{Y(t) : t \in U\}$ is h-localisable (resp. strongly h-localisable) at u. Let $a : U \to \mathbb{R}$ satisfy an η-Hölder condition on U, that is*

$$|a(t) - a(t')| \le c|t - t'|^\eta \quad (t, t' \in U),$$

where $\eta > h$. Then $aY = \{a(t)Y(t) : t \in U\}$ is h-localisable (resp. strongly h-localisable) at u with $(aY)'_u = a(u)Y'_u$.

Proof. We have

$$\frac{a(u+rt)Y(u+rt) - a(u)Y(u)}{r^h} = a(u+rt)\frac{Y(u+rt) - Y(u)}{r^h} + Y(u)\frac{a(u+rt) - a(u)}{r^h}.$$

The result now follows on letting $r \to 0$ with the appropriate form of convergence, noting that the right-hand term has zero limit almost surely. ■

4 Localisable processes with prescribed local form

We aim to construct localisable funtions with prescribed local form by 'joining together' localisable processes $\{X(t,v) : t \in U\}$ over a range of v. Thus we seek conditions that ensure $Y = \{X(t,t) : t \in U\}$ looks locally like $\{X(t,u) : t \in U\}$ when t is close to u.

Let U be an interval with u an interior point. Let $\{X(t,v) : (t,v) \in U \times U\}$ be a random field and let Y be the diagonal process $Y = \{X(t,t) : t \in U\}$. We want Y and

$X(\cdot,u)$ to have the same local forms at u, that is $Y'_u(\cdot) = X'_u(\cdot,u)$ where $X'_u(\cdot,u)$ is the local form of $X(\cdot,u)$ at u. Thus we require

$$\frac{X(u+rt,u+rt)-X(u,u)}{r^h} \stackrel{\text{fdd}}{\to} X'_u(t,u) \tag{4.1}$$

as $r \searrow 0$. The following theorem gives a sufficient condition for this to occur.

Theorem 4.1 *Let U be an interval with u an interior point. Suppose that for some $0<h<\eta$ the process $\{X(t,u): t\in U\}$ is h-localisable at $u\in U$ with local form $X'_u(\cdot,u)$ and*

$$\mathsf{P}(|X(v,v)-X(v,u)| \geq |v-u|^\eta) \to 0 \tag{4.2}$$

as $v\to u$. Then $Y=\{X(t,t): t\in U\}$ is h-localisable at u with $Y'_u(\cdot)=X'_u(\cdot,u)$.

In particular, this conclusion holds if for some $p>0$ and $\eta>h$

$$\mathsf{E}(|X(v,v)-X(v,u)|^p) = O(|v-u|^{\eta p}) \tag{4.3}$$

as $v\to u$.

Proof. For $r\neq 0$

$$\begin{aligned}\frac{Y(u+rt)-Y(u)}{r^h} &= \frac{X(u+rt,u+rt)-X(u,u)}{r^h}\\ &= \frac{X(u+rt,u+rt)-X(u+rt,u)}{r^h}+\frac{X(u+rt,u)-X(u,u)}{r^h}.\end{aligned} \tag{4.4}$$

Fix $t\in\mathbb{R}$ and $c>0$. Let r_0 be sufficiently small to ensure that if $0<r<r_0$ then both $u\pm rt\in U$ and $cr^h \geq (r|t|)^\eta$. Then for $0<r<r_0$

$$\begin{aligned}&\mathsf{P}\left(\frac{|X(u+rt,u+rt)-X(u+rt,u)|}{r^h}\geq c\right)\\ &\quad\leq \mathsf{P}\left(|X(u+rt,u+rt)-X(u+rt,u)|\geq (r|t|)^\eta\right)\\ &\quad\leq \mathsf{P}\left(|X(u+rt,u+rt)-X(u+rt,u)|\geq |(u+rt)-u|^\eta\right)\to 0\end{aligned}$$

as $r\searrow 0$, by (4.2). Thus for all $t\in\mathbb{R}$,

$$\frac{X(u+rt,u+rt)-X(u+rt,u)}{r^h}\to 0$$

in probability and so in finite dimensional distributions. Moreover,

$$\frac{X(u+rt,u)-X(u,u)}{r^h} \stackrel{\text{fdd}}{\to} X'_u(t,u),$$

since $X(\cdot,u)$ is localisable at u. We conclude from (4.4) that Y is localisable at u with local form $Y'_u(\cdot)=X'_u(\cdot,u)$.

If (4.3) holds, Markov's inequality implies (4.2) (with η replaced by some $h<\eta'<\eta$) and the conclusion follows. ■

Although Theorem 4.1 is valid for all $h>0$, it is normally applied with $0<h<1$. If $X(\cdot,u)$ is h-localisable for $h>1$ then the limit of (4.4) is usually dominated by the left-hand term giving that Y is 1-localisable, see Theorem 9.4 for an example of this.

5 Strongly localisable processes with prescribed local form

We obtain an analogue of Theorem 4.1 in the strongly localisable case, that is a criterion for convergence in distribution in (3.1).

Theorem 5.1 *Let $F(\mathbb{R})$ be either $C(\mathbb{R})$ endowed with the metric d or $D(\mathbb{R})$ with d_S, see (2.2) or (2.3). Let U be an interval with u an interior point. Suppose that for some $h > 0$ the process $\{X(t,u) : t \in U\}$ of $F(U)$ is strongly h-localisable at u, with local form $X'_u(\cdot, u)$ a random function of $F(\mathbb{R})$. Suppose that for all $c > 0$*

$$\mathsf{P}\left(\sup_{0<|v-u|<\epsilon} \frac{|X(v,v) - X(v,u)|}{|v-u|^h} > c \right) \to 0 \tag{5.1}$$

as $\epsilon \to 0$. If the process $Y = \{X(t,t) : t \in U\}$ is in $F(U)$ then Y is strongly h-localisable at u with $Y'_u(\cdot) = X'_u(\cdot, u)$.

In particular, this conclusion holds if for some $\eta > h$ we have

$$\sup_{v \in U, v \neq u} \frac{|X(v,v) - X(v,u)|}{|v-u|^\eta} < \infty \tag{5.2}$$

almost surely.

Proof. First consider $(C(\mathbb{R}), d)$. For each positive τ and r sufficiently small,

$$\begin{aligned}
&\mathsf{P}\left(\sup_{0<|t|\le\tau} \frac{|X(u+rt, u+rt) - X(u+rt, u)|}{r^h} > c \right) \\
&\qquad \le \mathsf{P}\left(\sup_{0<|t|\le\tau} \frac{|X(u+rt, u+rt) - X(u+rt, u)|}{|rt|^h} > \frac{c}{\tau^h} \right) \\
&\qquad \le \mathsf{P}\left(\sup_{0<|v-u|\le r\tau} \frac{|X(v,v) - X(v,u)|}{|v-u|^h} > \frac{c}{\tau^h} \right) \to 0
\end{aligned} \tag{5.3}$$

as $r \to 0$. Thus, the restriction of $\dfrac{X(u+rt, u+rt) - X(u+rt, u)}{r^h}$ to $[-\tau, \tau]$, converges to 0 in probability in $(C[-\tau,\tau], d^{[-\tau,\tau]})$ as $r \to 0$. From the definition (2.2) of d, convergence in probability on every bounded interval implies convergence in probability on $(C(\mathbb{R}), d)$, so

$$\frac{X(u+rt, u+rt) - X(u+rt, u)}{r^h} \xrightarrow{\mathrm{P}} 0 \tag{5.4}$$

in $(C(\mathbb{R}), d)$. Then

$$\begin{aligned}
\frac{Y(u+rt) - Y(u)}{r^h} &= \frac{X(u+rt, u+rt) - X(u,u)}{r^h} \\
&= \frac{X(u+rt, u+rt) - X(u+rt, u)}{r^h} + \frac{X(u+rt, u) - X(u,u)}{r^h} \\
&\xrightarrow{\mathrm{d}} X'_u(t,u),
\end{aligned} \tag{5.5}$$

as $r \to 0$, since X is localisable at u. Here we use a standard property [4, Theorem 4.1], that for random elements Z_r, Z, W_r, W of some metric space (M, ρ), if $Z_r \xrightarrow{\mathrm{d}} Z$ and $\rho(Z_r, W_r) \xrightarrow{\mathrm{p}} 0$, then $W_r \xrightarrow{\mathrm{d}} W$.

Turning to $(D(\mathbb{R}), d_S)$, if $X(t,u) \in D(\mathbb{R})$ and (5.1) holds, the same argument using (5.3) implies convergence in probability in (5.4) with respect to the metric d_S. (Note that $d_S^{[-\tau,\tau]}(f, 0) \leq \sup_{t\in[-\tau,\tau]} |f(t)|$ for each τ for $f \in D(\mathbb{R})$.) Convergence in distribution then follows just as for $C(\mathbb{R})$.

Finally, (5.1) is an immediate consequence of (5.2) if $h < \eta$. ∎

To utilise Theorem 5.1 we need to verify (5.2), that is to show that $Z(v) = (X(v,v) - X(v,u))/|v-u|^\eta$ is bounded as v ranges across an interval. The following form of Kolmogorov's continuity theorem will be extremely useful for this.

Theorem 5.2 *(Kolmogorov's continuity theorem) Let $\{Z(v) : v \in T\}$ be a random process where T is a bounded subset of $\mathbb{R}^n$. If for some $p > 0, \epsilon > 0$ and $c > 0$*

$$\mathsf{E}|Z(v) - Z(v')|^p \leq c|v - v'|^{n+\epsilon} \quad (v, v' \in T),$$

then Z has a continuous version that is almost surely η-Hölder continuous for all $0 < \eta < \epsilon/p$.

Proof. See, for example, [16, Theorem 25.2]. ∎

6 Multifractional Brownian motion with variable amplitude

A number of constructions of multifractional Brownian motion, a process with index-$h(u)$ fractional Brownian motion as its local form at u, have been given, see [1, 2, 3, 14]. Our method provides a straightforward construction of multifractional Brownian motion, that is strongly localisable with a given local index and amplitude.

As in [14] we model our definition on (3.2) but allow h to vary. By virtue of Proposition 3.2 variable local amplitude presents no difficulty. Let U be a bounded closed interval and let $h : U \to (0,1)$ satisfy an η-Hölder condition

$$|h(v) - h(v')| \leq k|v - v'|^\eta \quad (v, v' \in U) \tag{6.1}$$

where $0 < \eta \leq 1$. Note that by compactness there are numbers $0 < a \leq b < 1$ such that $h(u) \in [a,b]$ for all $u \in U$ (we shall often shrink intervals in this way without comment). We define a random field by the stochastic integral

$$X(t,v) = \int_{-\infty}^{\infty} \left((t-x)_+^{h(v)-1/2} - (-x)_+^{h(v)-1/2}\right) W(dx) \quad (t, v \in U), \tag{6.2}$$

where W is Wiener measure on $\mathbb{R}$. Since the integrand of (6.2) is square integrable, $X(t,v)$ exists a.s. with mean 0 for all $t, v \in U$.

We require the following estimate of the moments of increments.

Lemma 6.1 *Let $\{X(t,v) : t, u \in U\}$ be as in* (6.2)*. Then for all $p \geq 0$ there is a constant c such that*

$$\mathsf{E}|X(t,v) - X(t',v')|^p \leq c(|t-t'|^{pa} + |v-v'|^{p\eta}) \quad (t,t',v,v' \in U). \tag{6.3}$$

Proof. Firstly, since for fixed v' the process $X(\cdot, v')$ is just index-$h(v')$ fBm to within a constant $c(h(v'))$, see (3.2),

$$\mathsf{E}(X(t,v') - X(t',v'))^2 = c(h(v'))|t-t'|^{2h(v')} \leq c_1|t-t'|^{2a} \quad (t,t',v,v' \in U) \tag{6.4}$$

for some c_1. Secondly, applying the mean value theorem to the integrand of (6.2) for each $v \neq v', t \neq x$, we have

$$\begin{aligned} &X(t,v) - X(t,v') \\ &\quad = (h(v) - h(v')) \int_{-\infty}^{\infty} \Big((t-x)_+^{h(\cdot)-1/2} \log|t-x| - (-x)_+^{h(\cdot)-1/2} \log|-x| \Big) W(dx) \end{aligned}$$

where $h(\cdot) \equiv h(v,v',t,x) \in [h(v), h(v')]$. Thus

$$\begin{aligned} &\mathsf{E}(X(t,v) - X(t,v'))^2 \\ &\quad = (h(v) - h(v'))^2 \int_{-\infty}^{\infty} \Big((t-x)_+^{h(\cdot)-1/2} \log|t-x| - (-x)_+^{h(\cdot)-1/2} \log|-x| \Big)^2 dx \\ &\quad \leq c_2|v-v'|^{2\eta}, \end{aligned}$$

since by direct estimate or by comparison with the fBm integrals, this integral is uniformly bounded for $v, v', t \in U$, as $0 < a < h(v) < b < 1$ for some a, b. Combining with (6.4) this gives (6.3) when $p = 2$. But for each $p > 0$ there is a number $\rho_p > 0$ such that $E|Z|^p = \rho_p(E|Z|^2)^{p/2}$ for every zero mean Gaussian random variable Z, so (6.3) follows for all $p > 0$. ■

A first consequence of Lemma 6.1 is that the random field X has a continuous version.

Corollary 6.2 *With notation as above the random field X given by* (6.2) *has a continuous version that satisfies a Hölder condition of the following form: for all $\epsilon > 0$ there is an almost surely finite random constant C such that*

$$|X(t,v) - X(t',v')| \leq C(|t-t'|^{a-\epsilon} + |v-v'|^{\eta-\epsilon}) \quad (t,t',v,v' \in U). \tag{6.5}$$

Proof. Using (6.3) with p chosen sufficiently large, an argument similar to the usual derivation of Kolmogorov's continuity criterion, see [16, Theorem 25.2], gives (6.5). Alternatively it follows using a version of Dudley's metric entropy condition taking the metric

$$\rho((t,v),(t',v')) = (\mathsf{E}(X(t,v) - X(t',v'))^2)^{1/2} \leq c(|t-t'|^a + |v-v'|^\eta),$$

see [11, Section 15.4] ■

Strong localisability of mBm with varying local amplitude, defined by (6.6), now follows easily. The result for mBm was obtained using the harmonisable definition in [3], see also [14, Proposition 5].

Theorem 6.3 *(Multifractional Brownian motion) Let $u \in \mathbb{R}$ and let U be a closed interval with u an interior point. Suppose that $h : U \to (0,1)$ and $a : U \to \mathbb{R}^+$ both satisfy an η-Hölder condition where $h(u) < \eta \leq 1$. Define*

$$Y(t) = a(t) \int_{-\infty}^{\infty} \left((t-x)_+^{h(t)-1/2} - (-x)_+^{h(t)-1/2} \right) W(dx) \quad (t \in U). \tag{6.6}$$

Then Y is strongly $h(u)$-localisable at u with $Y'_u = a(u)c(h(u))B_{h(u)}$ where B_h is index-h fBm and where $c(h)$ is the normalisation constant in (3.2).

Proof. By Proposition 3.2 it is enough to consider the case where $a(v) \equiv 1$.

Let X be the random field (6.2). Choose $\epsilon > 0$ such that $h(u) < \eta - \epsilon$. By (6.5) there is an a.s. finite random variable C such that

$$|X(v,v) - X(v,u)| \leq C|v-u|^{\eta-\epsilon} \quad (v \in U)$$

so (5.2) holds (with η replaced by $\eta - \epsilon$). But $X(\cdot, u) = B_{h(u)}(\cdot)$ which is sssi so is strongly $h(u)$-localisable at u by Proposition 3.1. Theorem 5.1 implies that $Y = \{X(t,t) : t \in T\}$ is strongly $h(u)$-localisable at u with $Y'_u(\cdot) = X'_u(\cdot, u) = (B_{h(u)})'_u(\cdot) = B_{h(u)}(\cdot)$. ∎

7 Multifractional stable processes

Multifractional Brownian motion generalizes fractional Brownian motion by allowing the parameter h to vary with time. By working with a stochastic integral with respect to an α-stable measure instead of Wiener measure, we now construct multifractional stable processes with the local scaling exponent depending on t.

Recall that a process $\{X(t) : t \in T\}$, where T is generally a subinterval of $\mathbb{R}$, is called *α-stable* ($0 < \alpha \leq 2$) if all its finite-dimensional distributions are α-stable, see the encyclopaedic work on stable processes [17]. Note that 2-stable processes are just Gaussian processes.

Many stable processes admit a stochastic integral representation. Write $S_\alpha(\sigma, \beta, \mu)$ for the α-stable distribution with scale parameter σ, skewness β and shift-parameter μ; we will assume throughout that $\mu = 0$. Let $(E, \mathcal{E}, m)$ be a sigma-finite measure space (which will be Lebesgue measure in our examples). Taking m as the control measure and $\beta : E \to [-1,1]$ a measurable function, this defines an α-stable random measure M on E such that for $A \in \mathcal{E}$ we have that $M(A) \sim S_\alpha(m(A)^{1/\alpha}, \int_A \beta(x)m(dx)/m(A), 0)$. If $\beta = 0$ then the process is *symmetric* α-stable or SαS.

Let

$$\mathcal{F}_\alpha \equiv \mathcal{F}_\alpha(E, \mathcal{E}, m) = \{f : f \text{ is measurable and } \|f\|_\alpha < \infty\},$$

where $\| \, \|_\alpha$ is the quasinorm (or norm if $1 < \alpha \leq 2$) given by

$$\|f\|_\alpha = \begin{cases} \left(\int_E |f(x)|^\alpha m(dx) \right)^{1/\alpha} & (\alpha \neq 1) \\ \int_E |f(x)| m(dx) + \int_E |f(x)\beta(x) \ln |f(x)|| m(dx) & (\alpha = 1) \end{cases} \tag{7.1}$$

The stochastic integral of $f \in \mathcal{F}_\alpha(E, \mathcal{E}, m)$ with respect to M then exists [17, Chapter 3] with

$$I(f) = \int_E f(x) M(dx) \sim S_\alpha(\sigma_f, \beta_f, 0), \tag{7.2}$$

where

$$\sigma_f = \|f\|_\alpha, \quad \beta_f = \frac{\int f(x)^{<\alpha>}\beta(x)m(dx)}{\|f\|_\alpha^\alpha},$$

writing $a^{<b>} \equiv \mathrm{sign}(a)|a|^b$, see [17, Section 3.4]. In particular,

$$\mathsf{E}|I(f)|^p = \begin{cases} c(\alpha,\beta,p)\|f\|_\alpha^p & (0<p<\alpha) \\ \infty & (p \geq \alpha) \end{cases} \tag{7.3}$$

where $c(\alpha,\beta,p) < \infty$, see [17, Property 1.2.17].

When $0<\alpha<1$ there is a non-negative stable subordinator measure M' associated with M so that $M'(A) \sim S_\alpha(m(A)^{1/\alpha},1,0)$. In particular, for $f \in \mathcal{F}_\alpha$,

$$|I(f)| \leq \int_E |f(x)|M'(dx). \tag{7.4}$$

We will be concerned with processes that may be expressed as stochastic integrals

$$X(t) = \int_E f(t,x)M(dx) + \mu(t), \quad (t \in T), \tag{7.5}$$

where $f(t,\cdot)$ is a jointly measurable family of functions in $\mathcal{F}_\alpha(E,\mathcal{E},m)$ and $\mu(t)$ are real numbers. Note that if $\mathrm{esssup}_{a\leq t\leq b} f(t,x) = \infty$ for all $x \in A$ for some $A \subset E$ with $m(A) > 0$ then $X(t)$ will be unbounded a.s. on the interval $[a,b]$, see [17, Section 10].

Here we consider the localisibility at u of processes defined in terms of random fields

$$X(t,v) = \int_E f(t,v,x)M(dx) + \mu(t,v) \quad (t,v \in U) \tag{7.6}$$

where $f(t,v,.) \in \mathcal{F}_\alpha$ and $\mu(t,v) \in \mathbb{R}$ for all $t,v \in U$ for some interval U. We assume throughout that $f(t,v,x)$ is measurable on $U \times U \times E$.

The term $\mu(t,v)$ is easily dealt with: if $v \mapsto \mu(v,v)$ is pointwise η-Hölder at $v=u$, that is $|\mu(v,v)-\mu(u,u)| \leq k|u-v|^\eta$ for v close to u, where $0<h<\eta\leq 1$, then the h-localisability of $Y = \{X(t,t) : t \in U\}$ at u and its local form are unaffected if we set $\mu(t,v)=0$, so we assume this throughout this section.

The following proposition gives conditions for Y to have a continuous or bounded version, which is needed for strong localisability to be meaningful. Note that these sufficient conditions are geared towards our context; for other aspects see [17, Chapters 10,12].

Proposition 7.1 *Let U be a closed interval. Let X be a random field defined by*

$$X(t,v) = \int_E f(t,v,x)M(dx) \quad (t,v \in U) \tag{7.7}$$

where $f(t,v,\cdot) \in \mathcal{F}_\alpha$ are jointly measurable and M is an α-stable random measure with control measure m and measurable skewness.

(a) Let $0<\alpha<1$. If

$$\| \sup_{t,v\in U} |f(t,v,x)| \|_\alpha < \infty, \tag{7.8}$$

then the random field (7.7) has a bounded version.

If in addition $\{f(t,v,x) : x \in E\}$ *is an equiuniformly continuous family for* $t, v \in U$, *then* (7.7) *has a continuous version.*

(*b*) *Let* $1 < \alpha < 2$ *and* $1/\alpha < \eta \le 1$. *If*

$$\|f(t,v,\cdot) - f(t',v',\cdot)\|_\alpha \le k\,(|v-v'|^\eta + |t-t'|^\eta) \quad (t,t',v,v' \in U), \tag{7.9}$$

then $Y = \{X(t,t) : t \in U\}$ *has a continuous version for* $t \in U$, *satisfying an a.s.* β*-Hölder condition for all* $0 < \beta < (\eta\alpha - 1)/\alpha$.

Proof. (a) Since $0 < \alpha < 1$ there exists a stable subordinator measure M' associated with M, so that M' has control measure m and $M'(A) \sim S_\alpha(m(A)^{1/\alpha}, 1, 0)$. By (7.4), for $t, v \in U$,

$$|X(t,v)| \le \int |f(t,v,x)| M'(dx) \le \int \sup_{t,v\in U} |f(t,v,x)| M'(dx) \equiv Z,$$

where Z is an almost surely finite random variable by (7.8), so $X(t,v)$ is a.s. bounded for $t, v \in U$.

Now assume also the equicontinuity condition. Given $\epsilon > 0$ we may, since E is σ-finite, choose $D \subset E$ such that $\int_{E\setminus D} \left(\sup_{t,v\in U} |f(t,v,x)|\right)^\alpha m(dx) < \epsilon^\alpha$. By equiuniform continuity we may find $\delta > 0$ such that for all $x \in E$ and $|(t,v) - (t',v')| < \delta$ we have $|f(t,v,x) - f(t',v',x)| < m(D)^{-1/\alpha}\epsilon$. Then if $|(t,v) - (t',v')| < \delta$, (7.4) gives

$$\begin{aligned} |X(t,v) - X(t',v')| &\le \int_E |f(t,v,x) - f(t',v',x)| M'(dx) \\ &\le 2\int_{E\setminus D} \sup_{t,v\in U} |f(t,v,x)| M'(dx) + \int_D \frac{\epsilon}{m(D)^{1/\alpha}} M'(dx) \equiv Z_\epsilon, \end{aligned}$$

say, where Z_ϵ is a random variable. Fix $0 < p < \alpha$. By (7.3) there is a constant c independent of ϵ such that

$$\mathsf{E}|Z_\epsilon|^p \le c\epsilon^p.$$

Thus choosing $\epsilon(n)\,(n = 1, 2, \ldots)$ such that $\mathsf{E}|Z_{\epsilon(n)}|^p \le 2^{-n}$, there are corresponding δ_n such that

$$\sup_{|(t,v)-(t',v')|<\delta_n} |X(t,v) - X(t',v')| \le Z_{\epsilon(n)}.$$

Since $\sum_{n=1}^\infty \mathsf{E}|Z_{\epsilon(n)}|^p < \infty$, the Borel-Cantelli lemma gives that $Z_{\epsilon(n)} \to 0$ almost surely, so $\sup_{|(t,v)-(t',v')|<\delta_n} |X(t,v) - X(t',v')| \to 0$ a.s. as $n \to \infty$, giving continuity of $X(t,v)$ a.s.

(b) From (7.7)

$$X(t,v) - X(t',v') = \int (f(t,v,x) - f(t',v',x))\, M(dx).$$

This integrand is in $\mathcal{F}_\alpha$, so for $0 < p < \alpha$, estimate (7.3) gives

$$\begin{aligned} \mathsf{E}|X(t,v) - X(t',v')|^p &\le c_1 \|f(t,v,\cdot) - f(t',v',\cdot)\|_\alpha^p \\ &\le c_2\,(|v-v'|^{\eta p} + |t-t'|^{\eta p}) \end{aligned}$$

by (7.9) where c_1 and c_2 are independent of $t, t', v, v' \in U$. Specialising,

$$\mathsf{E}|Y(t) - Y(t')|^p = \mathsf{E}|X(t,t) - X(t',t')|^p \leq 2c_2|t-t'|^{\eta p}$$

for $t, t' \in U$.

Since $\eta > 1/\alpha$ we may choose $0 < p < \alpha$ such that $\eta p > 1$. Kolmogorov's Theorem 5.2 gives that Y has a continuous version for $t \in U$ with an a.s. β-Hölder condition for all $0 < \beta < (\eta p - 1)/p$ for all $p < \alpha$. ∎

We require the following calculus lemma.

Lemma 7.2 *Let U be an interval and let $f : U \to \mathbb{R}$ be continuously differentiable with f' satisfying an η-Hölder condition*

$$|f'(v) - f'(w)| \leq k|v-w|^\eta \quad (v, w \in U) \tag{7.10}$$

for some $0 < \eta \leq 1$. Let $v, w, u \in U$ with $v \neq u, w \neq u$. Then

$$\left| \frac{f(v)-f(u)}{v-u} - \frac{f(w)-f(u)}{w-u} \right| \leq 2^\eta k|v-w|^\eta. \tag{7.11}$$

Proof. We may assume without loss of generality that $v < w$ and $u < w$. Write $g(v) = \dfrac{f(v)-f(u)}{v-u}$. We consider three cases.

(a) If $v < u < w$, then by the mean value theorem there exist $v_0 \in (v,u)$ and $w_0 \in (u,w)$ such that $g(v) = f'(v_0)$ and $g(w) = f'(w_0)$. Then

$$|g(v) - g(w)| = |f'(v_0) - f'(w_0)| \leq k|v_0 - w_0|^\eta \leq k|v-w|^\eta.$$

(b) If $u < v < w$ and $|w-v| \geq |v-u|$, then $|w-v| \geq \frac{1}{2}|w-u|$. There exist $v_0 \in (u,v)$ and $w_0 \in (u,w)$ such that $g(v) = f'(v_0)$ and $g(w) = f'(w_0)$, so

$$|g(v) - g(w)| = |f'(v_0) - f'(w_0)| \leq k|v_0 - w_0|^\eta \leq k|w-u|^\eta \leq k2^\eta|w-v|^\eta.$$

(c) If $u < v < w$ and $|w-v| \leq |v-u|$, we apply the mean value theorem to g. Thus there exists $s \in (v,w)$ such that

$$\begin{aligned} g(v) - g(w) &= (v-w)g'(s) \\ &= (v-w)\frac{(s-u)f'(s) - f(s) + f(u)}{(s-u)^2} \\ &= (v-w)\frac{f'(s) - f'(z)}{(s-u)} \end{aligned}$$

where $z \in (u,s)$ using the mean value theorem again. Hence

$$\begin{aligned} |g(v) - g(w)| &\leq k\frac{|v-w||s-z|^\eta}{|s-u|} \\ &\leq k|v-w||s-u|^{\eta-1} \\ &\leq k|v-w||v-u|^{\eta-1} \\ &\leq k|v-w|^\eta. \end{aligned}$$

∎

The following theorem gives conditions that allow the transfer of localisability properties from $X(\cdot, u)$ to $Y = \{X(t,t) : t \in U\}$ in the α-stable case, generalising the results of Section 6 in the Gaussian case.

Theorem 7.3 *Let U be a closed interval with u an interior point. Let X be a random field defined by*

$$X(t,v) = \int f(t,v,x)M(dx) \quad (t,v \in U) \tag{7.12}$$

where $f(t,v,\cdot) \in \mathcal{F}_\alpha$ are jointly measurable and M is an α-stable random measure with control measure m and measurable skewness.

(*a*) *Suppose that $0 < \alpha \leq 2$ and the process $X(\cdot,u)$ is h-localisable at u with $h > 0$. Suppose that for some $\eta > h$*

$$\|f(t,v,\cdot) - f(t,u,\cdot)\|_\alpha \leq k_1|v-u|^\eta \quad (t,v \in U). \tag{7.13}$$

Then $Y = \{X(t,t) : t \in U\}$ is h-localisable at u with local form $Y'_u(\cdot) = X'_u(\cdot,u)$.

(*b*) *Suppose that $0 < \alpha < 1$ and that $X(\cdot,u)$ is strongly h-localisable in $C(\mathbb{R})$ (resp. $D(\mathbb{R})$) at u. Suppose that for some $\eta > h$*

$$|f(t,v,x) - f(t,u,x)| \leq k_1(x)|v-u|^\eta \quad (t,v \in U, x \in E), \tag{7.14}$$

where $k_1(\cdot) \in \mathcal{F}_\alpha$. If $Y = \{X(t,t) : t \in U\}$ has a version in $C(U)$ (resp. $D(U)$) (see Proposition 7.1(a)), then Y is strongly h-localisable at u in $C(\mathbb{R})$ (resp. $D(\mathbb{R})$) with $Y'_u(\cdot) = X'_u(\cdot,u)$.

(*c*) *Suppose that $1 < \alpha \leq 2$, that $\eta > 1/\alpha$ and that $X(\cdot,u)$ is strongly h-localisable in $C(\mathbb{R})$ or $D(\mathbb{R})$ at u. Suppose that for all $t,v \in U$ the partial derivative $f_v(t,v,\cdot) \in \mathcal{F}_\alpha$ with*

$$|f_v(t,v,x) - f_v(t,v',x)| \leq k_1(t,x)|v-v'|^\eta \quad (t,v,v' \in U, x \in E), \tag{7.15}$$

where $\sup_{t\in U}\|k_1(t,\cdot)\|_\alpha < \infty$, and that

$$\sup_{v\in U}|f_v(t,v,x) - f_v(t',v,x)| \leq k_2(t,t',x) \quad (t,t' \in U, x \in E), \tag{7.16}$$

where $\|k_2(t,t',\cdot)\|_\alpha \leq c|t-t'|^\eta$. Then $Y = \{X(t,t) : t \in U\}$ is strongly h-localisable at u in $C(\mathbb{R})$ with $Y'_u(\cdot) = X'_u(\cdot,u)$.

Proof. (a) We have

$$X(t,v) - X(t,u) = \int (f(t,v,x) - f(t,u,x))\, M(dx) \tag{7.17}$$

so, taking $0 < p < \alpha$ and using (7.3), there is a constant c_1 such that

$$\begin{aligned} \mathsf{E}|X(t,v) - X(t,u)|^p &\leq c_1 \|f(t,v,\cdot) - f(t,u,\cdot)\|_\alpha^p \\ &\leq c_1 k_1 |v-u|^{\eta p}. \end{aligned}$$

The conclusion follows from Theorem 4.1.

(b) Since $0 < \alpha < 1$ there exists a stable subordinator measure M' associated with M, so that M' has control measure m and $M'(A) \sim S_\alpha(m(A)^{1/\alpha}, 1, 0)$. Applying (7.4) to (7.17) and using (7.14), gives that for $t,v \in U$

$$\begin{aligned} |X(t,v) - X(t,u)| &\leq \int |f(t,v,x) - f(t,u,x)| M'(dx) \\ &\leq |v-u|^\eta \int k_1(x) M'(dx) \\ &\leq |v-u|^\eta Z, \end{aligned}$$

where Z is an a.s. finite random variable. Thus (5.2) holds and Theorem 5.1 gives that Y is strongly localisable at u.

(c) It is easy to check that Y satisfies the conditions of Theorem 7.1(b) and so has a continuous version. Again we verify (5.2). Define

$$\begin{aligned} Z(t,v) &= \frac{X(t,v)-X(t,u)}{v-u} \quad (t,v\in U, v\neq u) \\ &= \int g(t,v,x)M(dx) \end{aligned}$$

where

$$g(t,v,x) = \frac{f(t,v,x)-f(t,u,x)}{v-u}.$$

Applying Lemma 7.2 with $f(v)=f(t,v,x)$ and noting (7.15), we get

$$|g(t,v,x)-g(t,v',x)| \leq 2^{\eta}k_1(t,x)|v-v'|^{\eta}. \tag{7.18}$$

Also

$$\begin{aligned} |g(t,v,x)-g(t',v,x)| &= \frac{1}{|v-u|}|(f(t,v,x)-f(t',v,x))-(f(t,u,x)-f(t',u,x))| \\ &\leq |f_v(t,v_1,x)-f_v(t',v_1,x)| \\ &\leq k_2(t,t',x), \end{aligned} \tag{7.19}$$

for some $v_1 \in (u,v)$, on applying the mean value theorem to $f(t,v,x)-f(t',v,x)$. From (7.18) and (7.19) together with the conditions on k_1 and k_2 we get

$$\|g(t,v,\cdot)-g(t',v',\cdot)\|_{\alpha} \leq c_1(|v-v'|^{\eta}+|t-t'|^{\eta}).$$

Applying Proposition 7.1(b) to $\{Z(v,v): v\in U\}$, it follows that $Z(v,v)=(X(v,v)-X(v,u))/(v-u)$ has a version that is a.s. continuous and bounded for $v\in U$. Thus (5.2) holds and strong localisability follows from Theorem 5.1. ∎

We illustrate Theorem 7.3 by constructing processes whose local forms are linear stable fractional motions $L_{\alpha,h(t)}$, see (3.3). Overlapping results with a different emphasis are given in [19, 20]. The following process is termed a *linear stable multifractional motion*:

$$\begin{aligned} Y(t) = \int_{-\infty}^{\infty} \Big[& a\left((t-x)_+^{h(t)-1/\alpha}-(-x)_+^{h(t)-1/\alpha}\right) \\ & + b\left((t-x)_-^{h(t)-1/\alpha}-(-x)_-^{h(t)-1/\alpha}\right)\Big] M(dx) \quad (t\in\mathbb{R}), \end{aligned} \tag{7.20}$$

where M is an α-stable random measure $(0<\alpha<2)$ with constant skewness intensity β and control measure Lebesgue measure, with $h(t)\in(0,1)$ for all $t\in\mathbb{R}$, and a and b real numbers. (Recall that $(w)_+=\max\{0,w\}$ and $(w)_-=-(w)_+$ for $w\in\mathbb{R}$.)

To investigate localisability, we introduce the random field

$$\begin{aligned} X(t,v) = \int_{-\infty}^{\infty} \Big[& a\left((t-x)_+^{h(v)-1/\alpha}-(-x)_+^{h(v)-1/\alpha}\right) \\ & + b\left((t-x)_-^{h(v)-1/\alpha}-(-x)_-^{h(v)-1/\alpha}\right)\Big] M(dx) \quad (t,v\in\mathbb{R}). \end{aligned} \tag{7.21}$$

Then $X(t,v)$ is well-defined since since for each (t,v) the α-th power of the integrand is Lebesgue integrable. For each fixed v the process $X(\cdot,v)$ is just a linear stable fractional motion (3.3) so is $h(v)$-localisable, with $X'_u(\cdot,v) = L_{\alpha,h(v)}(\cdot)$ for all $u \in \mathbb{R}$. Provided that $h(v) > 1/\alpha$ it is in $C(\mathbb{R})$ and is strongly localisable.

Theorem 7.4 *(Linear multifractional stable motion) Let U be a closed interval with u an interior point. Let $0 < \alpha < 2$ and $h : U \to (0,1)$. Define $\{Y(t) : t \in U\}$ by (7.20) .*

(a) Assume that h satisfies a η-Hölder condition at u

$$|h(v) - h(u)| \leq k|v-u|^\eta \quad (v \in U)$$

where $h(u) < \eta \leq 1$. Then Y is $h(u)$-localisable at u with local form $Y'_u = L_{\alpha,h(u)}$.

(b) If $1 < \alpha < 2$ and h is differentiable with $1/\alpha < h(u) < 1$ and

$$|h'(v) - h'(v')| \leq k|v-v'|^\eta \quad (v, v' \in U) \tag{7.22}$$

where $1/\alpha < \eta \leq 1$, then Y is strongly $h(u)$-localisable at u with local form $Y'_u = L_{\alpha,h(u)}$.

Proof. For brevity of exposition we give the proof in the case of well-balanced linear multifractional stable motion, that is with $a = b = 1$ in (7.20) and (7.21); the general case is very similar. Thus we take

$$f(t,v,x) = |t-x|^{h(v)-1/\alpha} - |x|^{h(v)-1/\alpha}$$

in Theorem 7.3 (when $h(v) = 1/\alpha$ such expressions are interpreted as $\mathbf{1}_{[0,t]}(x)$ where $\mathbf{1}_{[0,t]}$ is an indicator function). Then $X(t,v) = \int f(t,v,x)M(dx)$ and $Y(t) = \int f(t,t,x)M(dx)$.

(a) By continuity, we may assume that U is a sufficiently small interval to ensure that $h(v) < \eta$ for all $v \in U$. Fix h_-, h_+ such that $0 < h_- < h(v) < h_+ < 1$ for all $v \in U$. Then for each $t, v, v', x \in U$ with $x \neq 0, x \neq t$, the mean value theorem gives

$$\begin{aligned} |f(t,v,x) &- f(t,u,x)| \\ &= \left| |t-x|^{h(\cdot)-1/\alpha} \log|t-x| - |x|^{h(\cdot)-1/\alpha} \log|x| \right| |h(v) - h(u)| \\ &\leq \left| |t-x|^{h(\cdot)-1/\alpha} \log|t-x| - |x|^{h(\cdot)-1/\alpha} \log|x| \right| k|v-u|^\eta, \end{aligned} \tag{7.23}$$

where $h(\cdot) \equiv h(t,v,x) \in [h(v), h(u)]$. But

$$k \left| |t-x|^{h(\cdot)-1/\alpha} \log|t-x| - |x|^{h(\cdot)-1/\alpha} \log|x| \right| \leq k_1(t,x)$$

for all $t \in U, x \in \mathbb{R}$, where

$$k_1(t,x) = \begin{cases} c_1 \max\left\{1, |t-x|^{h_- - 1/\alpha} + |x|^{h_- - 1/\alpha}\right\} & (|x| \leq 1 + 2\max_{t \in U}|t|) \\ c_2 |x|^{h_+ - 1/\alpha - 1} & (|x| > 1 + 2\max_{t \in U}|t|) \end{cases} \tag{7.24}$$

for appropriately chosen constants c_1 and c_2. Then $\int k_1(t,x)^\alpha dx$ is finite and uniformly bounded for $t \in U$, so as $X(\cdot,u)$ is $h(u)$-localisable at u, Theorem 7.3(a) gives that $Y = \{X(t,t) : t \in U\}$ is $h(u)$-localisable at u with local form $Y'_u(\cdot) = X'_u(\cdot,u) = L_{\alpha,h(u)}(\cdot)$.

(b) We may assume that U is small enough and h_-, h_+ are chosen so that $0 < 1/\alpha < h_- < h(v) < h_+ < 1$ for all $v \in U$. A similar estimate to (7.23) on the derivatives gives

$$\begin{aligned} |f_v(t,v,x) - f_v(t,v',x)| &\leq [|h'(v)||h'(v(\cdot))||v-v'| + |h(v) - h(v')|]\, k_1(t,x) \\ &\leq c_1 k_1(t,x)|v-v'|^\eta, \end{aligned} \tag{7.25}$$

for $t, v, v' \in U, x \in \mathbb{R}$, where $k_1(t,x)$ is as in (7.24), so (7.15) is satisfied. Moreover,

$$\begin{aligned} |f_v(t,v,x) &- f_v(t',v,x)| \\ &= |h'(v)|\big||t-x|^{h(v)-1/\alpha}\log|t-x| - |t'-x|^{h(v)-1/\alpha}\log|t'-x|\big| \\ &\leq k_2(t,t',x), \end{aligned} \tag{7.26}$$

for $t, t', v \in U, x \in \mathbb{R}$, where

$$k_2(t,t',x) = \begin{cases} c_2|t-t'|^{h_- -1/\alpha} & (|x-\frac{1}{2}(t-t')| \leq |t-t'|) \\ c_3|x-\frac{1}{2}(t-t')|^{h_+ -1/\alpha-1}|t-t'| & (|x-\frac{1}{2}(t-t')| > |t-t'|) \end{cases} \tag{7.27}$$

for constants c_2, c_3. Then $\|k_2(t,t',\cdot)\|_\alpha \leq c_4|t-t'|^{h_-}$, so (7.16) is satisfied taking $\eta = h_-$. Strong localisability follows from Theorem 7.3(c). ■

To conclude this section we examine stationary moving average processes. These provide examples of localisable α-stable processes of a rather different nature being stationary processes and not based on existing sssi processes.

Proposition 7.5 *Let $0 < \alpha \leq 2$, let $g \in \mathcal{F}_\alpha$ and let M be a symmetric α-stable measure on $\mathbb{R}$ with control measure $\mathcal{L}$. Define the stationary process Y by*

$$Y(t) = \int g(t-x)M(dx) \quad (t \in \mathbb{R}). \tag{7.28}$$

Suppose that there exist jointly measurable functions $h(t,.) \in \mathcal{F}_\alpha$ such that

$$\lim_{r\to 0}\int \left|\frac{g(r(t+z)) - g(rz)}{r^\gamma} - h(t,z)\right|^\alpha dz = 0 \tag{7.29}$$

for all $t \in \mathbb{R}$, where $\gamma + (1/\alpha) > 0$. Then Y is $(\gamma + (1/\alpha))$-localisable at all $u \in \mathbb{R}$ with local form $Y_u = \{\int h(t,z)M(dz) : t \in \mathbb{R}\}$.

Proof. Using stationarity followed by a change of variable $z = -x/r$ and the self-similarity of M,

$$\begin{aligned} Y(u+rt) - Y(u) &= Y(rt) - Y(0) \\ &= \int (g(rt-x) - g(-x))M(dx) \\ &= r^{1/\alpha}\int (g(r(t+z)) - g(rz))M(dz) \end{aligned}$$

where equality is in finite dimensional distributions. Thus

$$\frac{Y(u+rt) - Y(u)}{r^{\gamma+1/\alpha}} - \int h(t,z)M(dz) = \int \left(\frac{g(r(t+z)) - g(rz)}{r^\gamma} - h(t,z)\right) M(dz).$$

By [17, Proposition 3.5.1] and (7.29), $r^{-\gamma-1/\alpha}(Y(u+rt) - Y(u)) \to \int h(t,z)M(dz)$ in probability and thus in finite dimensional distributions. ■

A particular instance of (7.28) is the reverse Ornstein-Uhlenbeck process, see [17, Section 3.6].

Theorem 7.6 *(Reverse Ornstein-Uhlenbeck process) Let $\lambda > 0$ and $0 < \alpha \leq 2$ and let M be an α-stable measure on $\mathbb{R}$ with control measure $\mathcal{L}$. The stationary process defined by*

$$Y(t) = \int_t^\infty \exp(-\lambda(x-t))M(dx) \quad (t \in \mathbb{R})$$

has a version in $D(\mathbb{R})$ that is $1/\alpha$-localisable at all $u \in \mathbb{R}$ with $Y'_u = L_\alpha$, where L_α is α-stable Lévy motion.

Proof. The process Y is a stationary Markov process which has a version in $D(\mathbb{R})$ see [18, Remark 17.3]. It is a moving average process taking $g(x) = \exp(\lambda x)\mathbf{1}_{(-\infty,0]}(x)$ in (7.28). It is easily verified using the dominated convergence theorem that g satisfies (7.29) with $\gamma = 0$ and $h(t,z) = -\mathbf{1}_{[-t,0]}(z)$, so Proposition 7.5 gives the conclusion with $Y'_u(t) = -M([-t,0]) = L_\alpha(t)$. ∎

8 Sums over Poisson processes

In the next section we will set up 'multistable processes', that is α-stable processes where α is allowed to vary with t. For this it is convenient to express the random field $X(t,v)$ as a sum over a suitable Poisson point process.

In this section we bring together the basic properties of Poisson sums that we need. Let $(E, \mathcal{E}, m)$ be a σ-finite measure space. We work throughout with a Poisson point process Π on $E \times \mathbb{R}$, with mean measure $m \times \mathcal{L}$ where $\mathcal{L}$ is Lebesgue measure. Thus Π is a random countable subset of $E \times \mathbb{R}$ such that, writing $N(A)$ for the number of points in a measurable $A \subset E \times \mathbb{R}$, the random variable $N(A)$ has a Poisson distribution of mean $(m \times \mathcal{L})(A)$ with $N(A_1), \ldots, N(A_n)$ independent for disjoint $A_1, \ldots, A_n \subset \mathbb{R}^2$, see [12].

We define a quasinorm on certain spaces of measurable functions on E. For $0 < a \leq b < 2$ let

$$\mathcal{F}_{a,b} \equiv \mathcal{F}_{a,b}(E, \mathcal{E}, m) = \{f : f \text{ is } m\text{-measurable with } \|f\|_{a,b} < \infty\}$$

where

$$\|f\|_{a,b} = \left(\int_E |f(x)|^a m(dx)\right)^{1/a} + \left(\int_E |f(x)|^b m(dx)\right)^{1/b}. \tag{8.1}$$

(Of course $\| \, \|_{a,b}$ is a norm if $1 \leq a \leq b$.) Note that if $a \leq a' \leq b' \leq b$ then $\mathcal{F}_{a,b} \subset \mathcal{F}_{a',b'}$ and $\|f\|_{a',b'} \leq c\|f\|_{a,b}$ where c depends on a, a', b', b. Moreover, $\mathcal{F}_{a,a} = \mathcal{F}_a$.

The following estimate will be useful. Note that expressions such as (8.2) have two parts since we need to control the growth of $g(x,y)$ at both small and large values of y.

Lemma 8.1 *Let $g : E \times \mathbb{R} \to \mathbb{R}$ be $\mathcal{L}^2$-measurable and suppose that*

$$|g(x,y)| \leq h(x)\left(|y|^{-1/a} + |y|^{-1/b}\right) \tag{8.2}$$

where $h \in \mathcal{F}_{a,b}$ for some $0 < a \leq b < 2$. Then there is a constant c depending only on a and b such that

$$\int\int \sin^2(\tfrac{1}{2}\theta g(x,y))m(dx)dy \leq c\left(\theta^a \int |h(x)|^a m(dx) + \theta^b \int |h(x)|^b m(dx)\right) \quad (\theta \geq 0). \tag{8.3}$$

Proof. We have

$$\begin{aligned}
\int\int \sin^2(\tfrac{1}{2}\theta g(x,y))m(dx)dy &\leq \int\int \min\left\{\tfrac{1}{4}\theta^2|g(x,y)|^2,1\right\}m(dx)dy \\
&\leq c_1\int\int \min\left\{\theta^2|h(x)|^2|y|^{-2/a},1\right\}m(dx)dy \\
&\quad +c_1\int\int \min\left\{\theta^2|h(x)|^2|y|^{-2/b},1\right\}m(dx)dy, \qquad (8.4)
\end{aligned}$$

where c_1 is a constant, using (8.2) and making a simple estimate. But

$$\begin{aligned}
&\int\int \min\left\{\theta^2|h(x)|^2|y|^{-2/a},1\right\}m(dx)dy \\
&\qquad\leq \int\left[\int_{|y|\leq|\theta h(x)|^a} dy + \theta^2|h(x)|^2\int_{|y|>|\theta h(x)|^a}|y|^{-2/a}dy\right]m(dx) \\
&\qquad\leq c_2\theta^a\int|h(x)|^a m(dx)
\end{aligned}$$

where c_2 depends only on a, so along with a similar estimate with b replacing a, (8.4) gives (8.3). ∎

The next proposition gives criteria for the convergence of Poisson sums. We write (X,Y) for a random point of $E\times\mathbb{R}$ of the Poisson process Π.

Proposition 8.2 *Let $g: E\times\mathbb{R}\to\mathbb{R}$ be $m\times\mathcal{L}$-measurable with*

$$|g(x,y)|\leq h(x)\left(|y|^{-1/a}+|y|^{-1/b}\right) \tag{8.5}$$

where $h\in\mathcal{F}_{a,b}$.

(*a*) *If $0<a\leq b<1$ then the series*

$$\Sigma\equiv\sum_{(\mathsf{X},\mathsf{Y})\in\Pi} g(\mathsf{X},\mathsf{Y}) \tag{8.6}$$

converges absolutely almost surely.

(*b*) *Suppose that $0<a\leq b<2$ and that g is symmetric in the sense that*

$$g(x,-y)=-g(x,y) \qquad (x,y)\in E\times\mathbb{R}. \tag{8.7}$$

Let E_n be an increasing sequence of m-measurable subsets of E with $m(E_n)<\infty$ for all n and $\cup_{n=1}^{\infty}E_n=E$ and write R_n for the rectangle $\{(x,y): x\in E_n, |y|\leq n\}\subset E\times\mathbb{R}$. Then we may define

$$\Sigma\equiv\sum_{(\mathsf{X},\mathsf{Y})\in\Pi} g(\mathsf{X},\mathsf{Y})=\lim_{n\to\infty}\sum_{(\mathsf{X},\mathsf{Y})\in\Pi\cap R_n} g(\mathsf{X},\mathsf{Y}), \tag{8.8}$$

where the series converges almost surely.

(*c*) *Provided the symmetry condition (8.7) holds, the characteristic function of Σ, taking either definition (8.6) or definition (8.8), is given by*

$$\mathsf{E}(e^{i\theta\Sigma})=\exp\left(-2\int\int\sin^2(\tfrac{1}{2}\theta g(x,y))m(dx)dy\right) \quad (\theta\in\mathbb{R}). \tag{8.9}$$

Proof. If $0 < a \leq b < 1$, (8.5) easily implies that $\int \min\{|g(x,y)|, 1\} m(dx)dy < \infty$. By Campbell's theorem [12, Section 3.2] the random sum (8.6) is absolutely convergent almost surely with characteristic function

$$\mathsf{E}(e^{i\theta\Sigma}) = \exp\left(\int\int \left(e^{i\theta g(x,y)} - 1\right) m(dx)dy\right) \quad (\theta \in \mathbb{R}).$$

If the symmetry condition (8.7) holds, this reduces to (8.9).

In case (b) write $\Sigma_n = \sum_{(\mathsf{X},\mathsf{Y})\in\Pi\cap R_n} g(\mathsf{X},\mathsf{Y}) = \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} g(\mathsf{X},\mathsf{Y})\mathbf{1}_{R_n}(\mathsf{X},\mathsf{Y})$, where $\mathbf{1}_{R_n}$ is the indicator function of R_n. Then by (8.5) $\int \min\{|g(x,y)\mathbf{1}_{R_n}(x,y)|, 1\} m(dx)dy < \infty$, so using Campbell's theorem just as before

$$\begin{aligned}\mathsf{E}(e^{i\theta\Sigma_n}) &= \exp\left(-2\int\int \sin^2(\tfrac{1}{2}\theta g(x,y))\mathbf{1}_{R_n}(x,y) m(dx)dy\right)\\ &\to \exp\left(-2\int\int \sin^2(\tfrac{1}{2}\theta g(x,y)) m(dx)dy\right),\end{aligned}$$

as $n \to \infty$ for all θ, by monotone convergence. By (8.3) there is a number $c_1 > 0$ such that

$$1 \geq \exp\left(-2\int\int \sin^2(\tfrac{1}{2}\theta g(x,y)) m(dx)dy\right) \geq \exp(-2c_1|\theta|^a) \geq 1 - 2c_1|\theta|^a$$

for $|\theta| \leq 1$, using that $1 - e^{-x} \leq x$ if $x \geq 0$. Thus $\lim_{n\to\infty} \mathsf{E}(e^{i\theta\Sigma_n})$ exists for all θ and is continuous at $\theta = 0$, so by Lévy's continuity theorem [5, Section 10.6], Σ_n converges in distribution to a random variable Σ with characteristic function (8.9).

We may write

$$\lim_{n\to\infty} \sum_{(\mathsf{X},\mathsf{Y})\in\Pi\cap R_n} g(\mathsf{X},\mathsf{Y}) = \sum_{n=1}^{\infty} \sum_{(\mathsf{X},\mathsf{Y})\in\Pi\cap(R_n\setminus R_{n-1})} g(\mathsf{X},\mathsf{Y})$$

(taking $R_0 = \emptyset$), which is an infinite sum of independent random variables that converges in distribution, so by another theorem of Lévy [5, Chapter 12] it also converges almost surely. ∎

Proposition 8.3 *Let* $\Sigma = \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} g(\mathsf{X},\mathsf{Y})$ *be as in* (8.6) *or* (8.8) *where* $g(x,-y) = -g(x,y)$ *and*

$$|g(x,y)| \leq h(x)\left(|y|^{-1/a} + |y|^{-1/b}\right) \tag{8.10}$$

for some $h \in \mathcal{F}_{a,b}$. *Then for* $0 < p < a$,

$$\mathsf{E}|\Sigma|^p \leq c\|h\|_{a,b}^p, \tag{8.11}$$

where c *depends only on* a, b *and* p.

Proof. A simple calculation using distribution functions (see [4, p.47]) gives

$$\begin{aligned}
\mathsf{P}\{|\Sigma| \geq \lambda\} &\leq \frac{\lambda}{2}\int_{-2/\lambda}^{2/\lambda} \big(1-\mathsf{E}(\exp(i\theta\Sigma))\big)d\theta \\
&= \frac{\lambda}{2}\int_{-2/\lambda}^{2/\lambda} \Big(1-\exp\Big(-2\int\int \sin^2(\tfrac{1}{2}\theta g(x,y))m(dx)dy\Big)\Big)d\theta \\
&\leq \frac{\lambda}{2}\int_{-2/\lambda}^{2/\lambda} \Big(1-\exp\Big(-2c\big(\theta^a\int |h(x)|^a m(dx)+\theta^b\int |h(x)|^b m(dx)\big)\Big)\Big)d\theta \\
&\leq c\lambda\int_{-2/\lambda}^{2/\lambda} \Big(\theta^a\int |h(x)|^a m(dx)+\theta^b\int |h(x)|^b m(dx)\Big)\, d\theta \\
&\leq c_1\lambda^{-a}\int |h(x)|^a m(dx)+c_1\lambda^{-b}\int |h(x)|^b m(dx) \\
&\equiv \lambda^{-a}h_a+\lambda^{-b}h_b,
\end{aligned}$$

say, where c_1 depends only on a and b, using (8.9) and (8.3). Then

$$\begin{aligned}
\mathsf{E}|\Sigma|^p &= p\int_0^\infty \lambda^{p-1}\mathsf{P}(|\Sigma|\geq\lambda)d\lambda \\
&\leq p\int_0^\infty \lambda^{p-1}\min\{1,\lambda^{-a}h_a\}d\lambda + p\int_0^\infty \lambda^{p-1}\min\{1,\lambda^{-b}h_b\}d\lambda \\
&\leq p\int_0^{h_a^{1/a}} \lambda^{p-1}d\lambda + ph_a\int_{h_a^{1/a}}^\infty \lambda^{p-a-1}d\lambda + p\int_0^{h_b^{1/b}} \lambda^{p-1}d\lambda + ph_b\int_{h_b^{1/b}}^\infty \lambda^{p-b-1}d\lambda \\
&\leq c_2(h_a^{p/a}+h_b^{p/b}) \\
&\leq c\|h\|_{a,b}^p
\end{aligned}$$

where c_2, c depend on a, b and p. ■

We will sometimes need the following variant of Proposition 8.3.

Corollary 8.4 *Let $0<p<a<a_1<b_1<b<2$ and let $f_1, f_2 \in \mathcal{F}_{a,b}$. Let*

$$\Sigma \equiv \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} \big(f_1(\mathsf{X})\mathsf{Y}^{<-1/\alpha_1>} - f_2(\mathsf{X})\mathsf{Y}^{<-1/\alpha_2>}\big)\,,$$

where $a_1 \leq \alpha_1, \alpha_2 \leq b_1$. Then

$$\mathsf{E}|\Sigma|^p \leq c\|f_1-f_2\|_{a,b}^p + c\|f_2\|_{a,b}^p|\alpha_1-\alpha_2|^p$$

where c depends only on a, a_1, b, b_1 and p.

Proof. Since

$$\begin{aligned}
\Sigma &= \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} (f_1(\mathsf{X})-f_2(\mathsf{X}))\mathsf{Y}^{<-1/\alpha_1>} + f_2(\mathsf{X})\sum_{(\mathsf{X},\mathsf{Y})\in\Pi}\big(\mathsf{Y}^{<-1/\alpha_1>}-\mathsf{Y}^{<-1/\alpha_2>}\big) \\
&= \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} (f_1(\mathsf{X})-f_2(\mathsf{X}))\mathsf{Y}^{<-1/\alpha_1>} + f_2(\mathsf{X})\sum_{(\mathsf{X},\mathsf{Y})\in\Pi}(\alpha_1-\alpha_2)\mathsf{Y}^{<-1/\alpha>}\alpha^{-2}\log|\mathsf{Y}|
\end{aligned}$$

where $\alpha \in [\alpha_1, \alpha_2]$, using the mean value theorem, the corollary follows from Proposition 8.3. ■

Note that the introduction of a_1 and b_1 in Corollary 8.4 is necessitated by the 'log' term to ensure uniformity of the constant c.

9 Multistable processes

We now show how our approach may be used to construct multistable processes, that is processes where the local stability index varies. The development of this section mirrors that of Section 7, but depends heavily on the properties of Poisson sums derived in Section 8. We seek an analogue of Theorem 7.3 but with the local form Y'_u an $\alpha(u)$-stable process with $\alpha(u)$ depending on u.

We first define a random field analogous to (7.12), but where the stable random measure M is not allied to a particular value of α. Whilst it would be possible to set up a random measure that resembles an $\alpha(u)$-stable measure close to u, this would be technically quite complicated. We therefore favour an alternative approach, using a representation by sums over Poisson processes. In particular this permits $X(\cdot, v)$ to be specified using the same underlying Poisson process for different v.

As before $(E, \mathcal{E}, m)$ is a σ-finite measure space, and Π is a Poisson process on $E \times \mathbb{R}$ with mean measure $m \times \mathcal{L}$. In the case of constant α, with M an symmeetric α-stable random measure on E with control measure m and skewness 0, the stochastic integral (7.2) may be expressed as a Poisson process sum

$$I(f) = \int f(x)M(dx) \;=\; c(\alpha) \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} f(\mathsf{X})\mathsf{Y}^{<-1/\alpha>} \quad (0 < \alpha < 2), \tag{9.1}$$

with the sum taken in the sense of (8.6) or (8.8), and with

$$c(\alpha) = \left(2\alpha^{-1}\Gamma(1-\alpha)\cos(\tfrac{1}{2}\pi\alpha)\right)^{-1/\alpha}, \tag{9.2}$$

see [17, Section 3.12]. (As before $a^{<b>} = \mathrm{sign}(a)|a|^b$ and $\mathcal{L}$ is Lebesgue measure.)

Particularly relevant in (9.1) is that the stability index α occurs only as an exponent of Y, since the underlying Poisson process does not depend on α, so by varying this exponent we can vary the stability index. Thus the random field

$$X(t, v) = \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} f(t, v, \mathsf{X})\mathsf{Y}^{<-1/\alpha(v)>} \tag{9.3}$$

gives rise to a *multistable* process with varying α, of the form

$$Y(t) \equiv X(t, t) = \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} f(t, t, \mathsf{X})\mathsf{Y}^{<-1/\alpha(t)>}. \tag{9.4}$$

We first consider continuity and boundedness of the processes.

Proposition 9.1 *Let U be a closed interval. Let X be the random field defined by*

$$X(t, v) = \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} f(t, v, \mathsf{X})\mathsf{Y}^{<-1/\alpha(v)>} \quad (t, v \in U) \tag{9.5}$$

where $f(t, v, \cdot) \in \mathcal{F}_{a,b}$ are jointly measurable and $\alpha : U \to (a, b)$ is continuous.

(a) Suppose $0 < a < \alpha(v) < b < 1$ for $v \in U$. If

$$\sup_{t,v\in U} |f(t, v, x)| \le k(x), \tag{9.6}$$

where $k \in \mathcal{F}_{a,b}$, then $\{X(t,v) : t, v \in U\}$ has a bounded version.

If in addition $\{f(t,v,x) : x \in E\}$ is an equiuniformly continuous family for $t, v \in U$, then X has a continuous version.

(b) Suppose that $1 < a < \alpha(v) < b < 2$ for $v \in U$ and $1/a < \eta \leq 1$. Suppose that

$$|\alpha(v) - \alpha(v')| \leq k_1 |v - v'|^\eta \quad (v, v' \in U), \tag{9.7}$$

that

$$\sup_{t,v \in U} \|f(t,v,\cdot)\|_{a,b} < \infty, \tag{9.8}$$

and

$$\|f(t,v,\cdot) - f(t',v',\cdot)\|_{a,b} \leq k_2 \left(|v - v'|^\eta + |t - t'|^\eta\right) \quad (t, t', v, v' \in U). \tag{9.9}$$

Then $Y = \{X(t,t) : t \in U\}$ has a continuous version satisfying an a.s. β-Hölder condition for all $0 < \beta < (\eta a - 1)/a$.

Proof. (a) From (9.5), for all $t, v \in U$,

$$\begin{aligned} |X(t,v)| &\leq \sum_{(\mathsf{X},\mathsf{Y}) \in \Pi} |f(t,v,\mathsf{X})| |\mathsf{Y}|^{-1/\alpha(v)} \\ &\leq \sum_{(\mathsf{X},\mathsf{Y}) \in \Pi} \sup_{t,v \in U} |f(t,v,\mathsf{X})| (|\mathsf{Y}|^{-1/a} + |\mathsf{Y}|^{-1/b}) \equiv Z \end{aligned}$$

where Z is an a.s. finite random variable, by Proposition 8.2(a). Thus $\{X(t,v) : t, v \in U\}$ is a.s. bounded.

Assuming also the equicontinuity condition, given $\epsilon > 0$ we may choose $r \geq 1$ such that $\|k(x)\mathbf{1}_{\{|x|>r\}}(x)\|_{a,b} < \epsilon$, where I is the indicator function. By equiuniform continuity we may find $\delta > 0$ such that for all $x \in \mathbb{R}$ and $|(t,v) - (t',v')| < \delta$ we have $|f(t,v,x) - f(t',v',x)| < r^{-1/a}\epsilon$, and $|\alpha(v) - \alpha(v')| < \epsilon$. Then if $|(t,v) - (t',v')| < \delta$, making several estimates in the obvious way,

$$\begin{aligned} |X(t,v) \;-\; X(t',v')| \;\leq\; & \sum_{(\mathsf{X},\mathsf{Y}) \in \Pi} |f(t,v,\mathsf{X})\mathsf{Y}^{<-1/\alpha(v)>} - f(t',v',\mathsf{X})\mathsf{Y}^{<-1/\alpha(v')>}| \\ \leq\; & \sum_{|\mathsf{X}| \leq r} |f(t,v,\mathsf{X}) - f(t',v',\mathsf{X})| |\mathsf{Y}|^{-1/\alpha(v)} + 2 \sum_{|\mathsf{X}| > r} \sup_{t,v \in U} |f(t,v,\mathsf{X})| |\mathsf{Y}|^{-1/\alpha(v)} \\ & + \sum_{(\mathsf{X},\mathsf{Y}) \in \Pi} |f(t',v',\mathsf{X})| |\mathsf{Y}^{<-1/\alpha(v)>} - \mathsf{Y}^{<-1/\alpha(v')>}| \\ \leq\; & \sum_{(\mathsf{X},\mathsf{Y}) \in \Pi} r^{-1/a}\epsilon \mathbf{1}_{\{|x| \leq r\}}(\mathsf{X}) |\mathsf{Y}|^{-1/\alpha(v)} + 2 \sum_{(\mathsf{X},\mathsf{Y}) \in \Pi} k(\mathsf{X}) \mathbf{1}_{\{|x|>r\}}(\mathsf{X}) |\mathsf{Y}|^{-1/\alpha(v)} \\ & + \sum_{(\mathsf{X},\mathsf{Y}) \in \Pi} |f(t',v',\mathsf{X})| \frac{1}{\alpha^2} |\mathsf{Y}|^{-1/\alpha} |\log |\mathsf{Y}|| \, |\alpha(v) - \alpha(v')| \\ \leq\; & \Big(\sum_{(\mathsf{X},\mathsf{Y}) \in \Pi} r^{-1/a}\epsilon \mathbf{1}_{\{|x| \leq r\}}(\mathsf{X}) + 2 \sum_{(\mathsf{X},\mathsf{Y}) \in \Pi} k(\mathsf{X}) \mathbf{1}_{\{|x|>r\}}(\mathsf{X}) \\ & + \sum_{(\mathsf{X},\mathsf{Y}) \in \Pi} c_1 |k(\mathsf{X})| \frac{1}{a^2}\epsilon \Big) (|\mathsf{Y}|^{-1/a} + |\mathsf{Y}|^{-1/b}) \equiv Z_\epsilon \end{aligned}$$

where Z_ϵ is a random variable, and we have used the mean value theorem in the third term of the sum with $\alpha \in [\alpha(v), \alpha(v')]$. Fix $0 < p < \alpha$. By (8.11) there is a constant c independent of ϵ such that

$$\mathsf{E}|Z_\epsilon|^p \leq c\epsilon^p.$$

The proof is completed just as in the proof of Proposition 7.1(a).

(b) We estimate

$$X(t,v) - X(t',v') = (X(t,v) - X(t,v')) + (X(t,v') - X(t',v')) \quad (t,t',v,v' \in U) \tag{9.10}$$

by considering its two parts in turn. Firstly

$$X(t,v) - X(t,v') = \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} \Big(f(t,v,\mathsf{X})\mathsf{Y}^{<-1/\alpha(v)>} - f(t,v',\mathsf{X})\mathsf{Y}^{<-1/\alpha(v')>} \Big)$$

Thus Corollary 8.4 gives, for $0 < p < a$,

$$\begin{aligned} \mathsf{E}|X(t,v) - X(t,v')|^p &\leq c_1\|f(t,v,\cdot) - f(t,v',\cdot)\|_{a,b}^p + c_1\|f(t,v',\cdot)\|_{a,b}^p|\alpha(v) - \alpha(v')|^p \\ &\leq c_2|v - v'|^{\eta p} \end{aligned} \tag{9.11}$$

by (9.9), (9.8) and (9.7).

For the second term of (9.10)

$$X(t,v) - X(t',v) = \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} (f(t,v,\mathsf{X}) - f(t',v,\mathsf{X}))\,\mathsf{Y}^{<-1/\alpha(v)>}.$$

Then

$$|(f(t,v,x) - f(t',v,x))y^{<-1/\alpha(v)>}| \leq |f(t,v,x) - f(t',v,x)| \left(|y|^{-1/a} + |y|^{-1/b}\right)$$

so, for $0 < p < a$, Proposition 8.3 and (9.9) give

$$\begin{aligned} \mathsf{E}|X(t,v) - X(t',v)|^p &\leq c_3\|f(t,v,\cdot) - f(t',v,\cdot)\|_{a,b}^p \\ &\leq c_4|t - t'|^{\eta p}. \end{aligned}$$

Combining with (9.11) we estimate (9.10) to get, for $t,t',v,v' \in U$,

$$\mathsf{E}|X(t,v) - X(t',v')|^p \leq c_5(|v - v'|^{\eta p} + |t - t'|^{\eta p}).$$

Specialising,

$$\mathsf{E}|Y(t) - Y(t')|^p = \mathsf{E}|X(t,t) - X(t',t')|^p \leq 2c_5|t - t'|^{\eta p}$$

for $t,t' \in U$.

Since $\eta > 1/a$ we may choose $0 < p < a$ such that $\eta p > 1$. Kolmogorov's Theorem 5.2 gives that $\{Y(t) : t \in U\}$ has a continuous version that is a.s. β-Hölder for all $0 < \beta < (\eta p - 1)/p$ for all $p < a$. ■

We come to the main result on the localisability of processes with varying stability index.

Theorem 9.2 *Let U be a closed interval with u an interior point and let $0 < a < b < 2$. Let X be the random field defined by*

$$X(t,v) = \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} f(t,v,\mathsf{X})\mathsf{Y}^{<-1/\alpha(v)>} \quad (t,v\in U) \tag{9.12}$$

where $f(t,v,\cdot)\in\mathcal{F}_{a,b}$ are jointly measurable and $\alpha : U \to (a,b)$.

(a) Suppose $X(\cdot,u)$ is h-localisable at u for $h>0$. Suppose that $\sup_{t\in U}\|f(t,u,\cdot)\|_{a,b} < \infty$, and that for some $\eta > h$

$$|\alpha(v)-\alpha(u)| \le k_1|v-u|^\eta \quad (v\in U), \tag{9.13}$$

and

$$\|f(t,v,\cdot)-f(t,u,\cdot)\|_{a,b} \le k_2|v-u|^\eta \quad (t,v\in U). \tag{9.14}$$

Then $Y=\{X(t,t):t\in U\}$ is h-localisable at u with local form $Y'_u(\cdot)=X'_u(\cdot,u)$.

(b) Suppose that $0<\alpha(u)<1$ and that $X(\cdot,u)$ is strongly h-localisable in $C(\mathbb{R})$ (resp. $D(\mathbb{R})$) at u. Suppose that for some $\eta>h$

$$|\alpha(v)-\alpha(u)| \le k_1|v-u|^\eta \quad (v\in U), \tag{9.15}$$

and

$$|f(t,u,x)| \le k_2(x) \quad (t\in U, x\in E), \tag{9.16}$$

and

$$|f(t,v,x)-f(t,u,x)| \le k_3(x)|v-u|^\eta \quad (t,v\in U, x\in E), \tag{9.17}$$

where $k_2(\cdot),k_3(\cdot)\in\mathcal{F}_{a,b}$. If $Y=\{X(t,t):t\in U\}$ has a version in $C(U)$ (resp. $D(U)$) then Y is strongly h-localisable in $C(\mathbb{R})$ (resp. $D(\mathbb{R})$) at u with $Y'_u(\cdot)=X'_u(\cdot,u)$.

(c) Suppose that $1<\alpha(u)<2$ and that $X(\cdot,u)$ is strongly h-localisable in $C(\mathbb{R})$ (resp. $D(\mathbb{R})$) at u. Let η satisfy $1/\alpha(u)<\eta\le 1$. Suppose that α is continuously differentiable on U with

$$|\alpha'(v)-\alpha'(v')| \le k_1|v-v'|^\eta \quad (v,v'\in U). \tag{9.18}$$

Suppose that the partial derivatives $f_v(t,v,\cdot)\in\mathcal{F}_{a,b}$ for all $t,v\in U$, and the following estimates hold:

$$\sup_{t\in U}\|f(t,u,\cdot)\|_{a,b}<\infty, \qquad \sup_{t,v\in U}\|f_v(t,v,\cdot)\|_{a,b}<\infty, \tag{9.19}$$

$$\|f(t,v,\cdot)-f(t',v,\cdot)\|_{a,b} \le k_2|t-t'|^\eta \quad (t,t',v\in U), \tag{9.20}$$

$$|f_v(t,v,x)-f_v(t,v',x)| \le k_3(t,x)|v-v'|^\eta \quad (t,v,v'\in U, x\in E), \tag{9.21}$$

and

$$|f_v(t,v,x)-f_v(t',v,x)| \le k_4(t,t',x) \quad (t,t',v\in U, x\in E), \tag{9.22}$$

where $\sup_{t\in U}\|k_3(t,\cdot)\|_{a,b}<\infty$, and $\|k_4(t,t',\cdot)\|_{a,b}\le k|t-t'|^\eta$ for all $t,t'\in U$. Then $Y=\{X(t,t):t\in U\}$ is strongly h-localisable in $C(\mathbb{R})$ at u with $Y'_u(\cdot)=X'_u(\cdot,u)$.

Proof. (a) We have

$$X(t,v)-X(t,u) = \sum_{(\mathsf{X},\mathsf{Y})\in\Pi}\left(f(t,v,\mathsf{X})\mathsf{Y}^{<-1/\alpha(v)>} - f(t,u,\mathsf{X})\mathsf{Y}^{<-1/\alpha(u)>}\right). \tag{9.23}$$

With $0 < p < a$, Corollary 8.4 gives that there are constants c_1, c_2 such that

$$\begin{aligned}\mathsf{E}|X(t,v)-X(t,u)|^p &\leq c_1\|f(t,v,\cdot)-f(t,u,\cdot)\|_{a,b}^p + c_1|\alpha(v)-\alpha(u)|^p\|f(t,u,\cdot)\|_{a,b}^p \\ &\leq c_2|v-u|^{\eta p},\end{aligned}$$

for all $t, v \in U$, by (9.13) and (9.14). Part (a) now follows from Theorem 4.1.

(b) We may assume that $a < \alpha(v) < b < 1$ for $v \in U$, if necessary using the continuity of α to replace U by a subinterval to decrease the value of b. Splitting and estimating (9.23), using the mean value theorem as in the proof of Corollary 8.4, we get, with $\alpha' \in [\alpha(v), \alpha(u)]$ (where α' depends on v),

$$\begin{aligned}|X(t,v)-X(t,u)| &\leq \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} |f(t,v,\mathsf{X})-f(t,u,\mathsf{X})||\mathsf{Y}|^{-1/\alpha(v)} \\ &\quad + \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} |f(t,u,\mathsf{X})||\alpha(v)-\alpha(u)||\mathsf{Y}|^{-1/\alpha'}\alpha'^{-2}|\log|\mathsf{Y}|\,| \\ &\leq |v-u|^\eta \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} |k_3(\mathsf{X})|(|\mathsf{Y}|^{-1/a}+|\mathsf{Y}|^{-1/b}) \\ &\quad + c_1|v-u|^\eta \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} |k_2(\mathsf{X})|(|\mathsf{Y}|^{-1/a}+|\mathsf{Y}|^{-1/b})\end{aligned}$$

for all $t, v \in U$, using (9.15)-(9.17). By Proposition 8.2(a) $|k_2(\mathsf{X})|(|\mathsf{Y}|^{-1/a}+|\mathsf{Y}|^{-1/b})$ and $|k_3(\mathsf{X})|(|\mathsf{Y}|^{-1/a}+|\mathsf{Y}|^{-1/b})$ are a.s. finite random variables, so (5.2) holds and Theorem 5.1 implies that Y is strongly localisable at u.

(c) The conditions of Proposition 9.1(b) are easily checked, so Y has a continuous version. We may assume that $1 < a < \alpha(v) < b$ for $v \in U$ and that $1/a < \eta$, using continuity of α to replace U by a subinterval and to increase the value of a if necesssary.

Define

$$Z(t,v) = \frac{X(t,v)-X(t,u)}{v-u} \quad (t, v \in U, v \neq u);$$

again we use Kolmogorov's criterion to show that $\{Z(v,v) : v \in U\}$ is almost surely bounded to get (5.2). We write

$$Z(t,v) = Z_1(t,v) + Z_2(t,v) \quad (t, v \in U) \tag{9.24}$$

where

$$Z_1(t,v) = \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} g(t,v,\mathsf{X})\mathsf{Y}^{<-1/\alpha(v)>} \text{ with } g(t,v,x) = \frac{f(t,v,x)-f(t,u,x)}{v-u} \tag{9.25}$$

and

$$Z_2(t,v) = \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} f(t,u,\mathsf{X})\frac{\mathsf{Y}^{<-1/\alpha(v)>}-\mathsf{Y}^{<-1/\alpha(u)>}}{v-u}. \tag{9.26}$$

For $p < a$ we estimate $\mathsf{E}|Z(t,v)-Z(t',v')|^p$ by breaking it into four parts.

(i) Applying Lemma 7.2 to (9.21) gives $|g(t,v,x)-g(t,v',x)| \leq 2^\eta k_3(t,x)|v-v'|^\eta$. Thus Corollary 8.4 on (9.25) and then (9.19) with the mean value theorem gives

$$\begin{aligned}\mathsf{E}|Z_1(t,v)-Z_1(t,v')|^p &\leq c_1\|g(t,v,\cdot)-g(t,v',\cdot)\|_{a,b}^p + c_1\|g(t,v',\cdot)\|_{a,b}^p|\alpha(v)-\alpha(v')|^p \\ &\leq c_2\|k_3(t,\cdot)\|_{a,b}^p|v-v'|^{\eta p} + c_2\sup_{t,v\in U}\|f_v(t,v,\cdot)\|_{a,b}^p|v-v'|^{\eta p} \\ &\leq c_3|v-v'|^{\eta p}. \end{aligned} \tag{9.27}$$

(ii) Using the mean value theorem and (9.22)

$$\begin{aligned}|g(t,v,x)-g(t',v,x)| &= \frac{1}{|v-u|}\,|(f(t,v,x)-f(t',v,x))-(f(t,u,x)-f(t',u,x))| \\ &= |f_v(t,v_1,x)-f_v(t',v_1,x)| \\ &\le k_4(t,t',x)\end{aligned}$$

where $v_1 \in (v,u)$ depends on t, t', v and x. Proposition 8.3 with (9.25) now gives

$$\mathsf{E}|Z_1(t,v)-Z_1(t',v)|^p \le c_4\|k_4(t,t',\cdot)\|_{a,b}^p \le c_5|t-t'|^{\eta p}. \tag{9.28}$$

(iii) Turning to (9.26), a simple estimate using (9.18) gives that

$$\left|\left[\frac{d}{dv}y^{<-1/\alpha(v)>}\right]_{v'}^{v}\right| \le c_6|v-v'|^\eta\left(|y|^{-1/a}+|y|^{-1/b}\right).$$

By Lemma 7.2

$$\left|\frac{y^{<-1/\alpha(v)>}-y^{<-1/\alpha(u)>}}{v-u}-\frac{y^{<-1/\alpha(v')>}-y^{<-1/\alpha(u)>}}{v'-u}\right| \le 2^\eta c_6|v-v'|^\eta\left(|y|^{-1/a}+|y|^{-1/b}\right).$$

Thus Proposition 8.3 applied to (9.26) gives

$$\mathsf{E}|Z_2(t,v)-Z_2(t,v')|^p \le c_7\|f(t,u,\cdot)\|_{a,b}^p|v-v'|^{\eta p} \le c_8|v-v'|^{\eta p}. \tag{9.29}$$

(iv) Finally, a further mean value estimate gives

$$\begin{aligned}&\left|(f(t,v,x)-f(t',v,x))\frac{y^{<-1/\alpha(v)>}-y^{<-1/\alpha(u)>}}{v-u}\right| \\ &\qquad\le c_9|(f(t,v,x)-f(t',v,x))|\left(|y|^{1/a}+|y|^{1/b}\right).\end{aligned}$$

By Proposition 8.3 and (9.20)

$$\mathsf{E}|Z_2(t,v)-Z_2(t',v)|^p \le c_{10}\|f(t,v,\cdot)-f(t',v,\cdot)\|_{a,b}^p \le c_{11}|t-t'|^{\eta p}. \tag{9.30}$$

Taking (9.24) with (9.27), (9.28), (9.29) and (9.30), we conclude that for some c_{12} independent of $t, t', v, v' \in U$,

$$\mathsf{E}|Z(t,v)-Z(t',v')|^p \le c_{12}(|v-v'|^{\eta p}+|t-t'|^{\eta p}) \tag{9.31}$$

if $0 < p < a$. Specialising,

$$\mathsf{E}|Z(v,v)-Z(v',v')|^p \le 2c_{12}|v-v'|^{\eta p} \tag{9.32}$$

for $v, v' \in U$.

Since $\eta > 1/a$ we may choose $0 < p < a$ such that $\eta p > 1$. Applying Kolomogorov's Theorem 5.2 to $\{Z(v,v) : v \in U\}$ we conclude that $Z(v,v) = \dfrac{X(v,v)-X(v,u)}{v-u}$ has a version that is a.s. bounded. Thus (5.2) holds and strong localisability follows from Theorem 5.1. ■

We now show how Theorem 9.2 may be used to construct some specific multistable processes. In these examples we take $(E, \mathcal{E}, m)$ to be Lebesgue measure on $\mathbb{R}$ so that Π is the Poisson process on $\mathbb{R}^2$ with mean measure $\mathcal{L}^2$. We first construct a multistable analogue of the linear multifractional motion of Theorem 7.4.

Theorem 9.3 *(Linear multistable multifractional motion). Let $a : \mathbb{R} \to \mathbb{R}^+$ be C^1 and $\alpha : \mathbb{R} \to (0,2)$ and $h : \mathbb{R} \to (0,1)$ be C^2. Define*

$$Y(t) = a(t)c(\alpha(t)) \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} \mathsf{Y}^{<-1/\alpha(t)>} \left(|t-\mathsf{X}|^{h(t)-1/\alpha(t)} - |\mathsf{X}|^{h(t)-1/\alpha(t)}\right) \quad (t \in \mathbb{R}). \quad (9.33)$$

(a) The process Y is $h(u)$-localisable at all $u \in \mathbb{R}$, with $Y'_u = a(u)L_{\alpha(u),h(u)}$, where $L_{\alpha,h}$ is linear stable motion.

(b) If u is such that $h(u) > 1/\alpha(u)$ then Y is strongly $h(u)$-localisable in $C(\mathbb{R})$ at u, with $Y'_u = a(u)L_{\alpha(u),h(u)}$.

Proof. By the amplitude result, Proposition 3.2, the term $a(t)c(\alpha(t))$ in (9.33) does not affect localisability, so it is enough to prove the result with $a(t)c(\alpha(t)) = 1$. Define a random field by

$$\begin{aligned} X(t,v) &= \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} \mathsf{Y}^{<-1/\alpha(v)>} \left(|t-\mathsf{X}|^{h(v)-1/\alpha(v)} - |\mathsf{X}|^{h(v)-1/\alpha(v)}\right) \quad (t,v \in \mathbb{R}) \\ &= \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} f(t,v,\mathsf{X})\mathsf{Y}^{<-1/\alpha(v)>}, \end{aligned}$$

where

$$f(t,v,x) = \left(|t-x|^{h(v)-1/\alpha(v)} - |x|^{h(v)-1/\alpha(v)}\right).$$

Then

$$\begin{aligned} &f_v(t,v,x) \\ &\quad = \left(|t-x|^{h(v)-1/\alpha(v)} \log|t-x| - |x|^{h(v)-1/\alpha(v)} \log|x|\right) \left(h'(v) + \alpha'(v)/\alpha(v)^2\right). \end{aligned}$$

Given $u \in \mathbb{R}$ we may use continuity of h and α to choose U to be a small enough closed interval with u an interior point, and numbers a, b, h_-, h_+, such that $0 < a < \alpha(v) < b < 2$ and $0 < h_- < h(v) < h_+ < 1$ for all $v \in U$, and such that $\frac{1}{a} - \frac{1}{b} < h_- < h_+ < 1 - (\frac{1}{a} - \frac{1}{b})$. A similar argument to that of Theorem 7.4 gives that

$$|f(t,v,x)|, |f_v(t,v,x)| \le k_1(t,x) \quad (t,v \in U, x \in \mathbb{R}) \quad (9.34)$$

and

$$|f(t,v,x) - f(t,v',x)|, |f_v(t,v,x) - f_v(t,v',x)| \le k_1(t,x)|v-v'| \, (t,v,v' \in U, x \in \mathbb{R}) \quad (9.35)$$

where

$$k_1(t,x) = \begin{cases} c_1 \max\{1, |t-x|^{h_- - 1/a} + |x|^{h_- - 1/a}\} & (|x| \le 1 + 2\max_{t\in U}|t|) \\ c_2|x|^{h_+ - 1/b - 1} & (|x| > 1 + 2\max_{t\in U}|t|) \end{cases} \quad (9.36)$$

for appropriately chosen constants c_1 and c_2. By virtue of the conditions on a, b, h_-, h_+ it follows that $\sup_{t\in U} \|k_1(t,\cdot)\|_{a,b} < \infty$. Since $X(\cdot,u) = c(\alpha(u))^{-1}L_{\alpha(u),h(u)}(\cdot)$, Theorem 9.2(a) gives $h(u)$-localisability of Y with $Y'_u(\cdot) = X'_u(\cdot,u) = c(\alpha(u))^{-1}(L_{\alpha(u),h(u)})'_u(\cdot) = c(\alpha(u))^{-1}L_{\alpha(u),h(u)}(\cdot)$.

For part (b), we choose U and the numbers a, b, h_-, h_+ to satisfy the conditions stipulated in the proof of (a) but also to satisfy $h_- > 1/a + (1/a - 1/b) > 0$, so in particular $h(v) - 1/\alpha(v') > 0$ for all $v, v' \in U$. Again as in Theorem 7.4,

$$|f(t,v,x) - f(t',v,x)|, |f_v(t,v,x) - f_v(t',v,x)| \leq k_2(t,t',x)$$

for $t, v \in U, x \in \mathbb{R}$, where

$$k_2(t,t',x) = \begin{cases} c_3|t-t'|^{h_- - 1/a} & (|x - \frac{1}{2}(t-t')| \leq |t-t'|) \\ c_4|x - \frac{1}{2}(t-t')|^{h_+ - 1/b - 1}|t-t'| & (|x - \frac{1}{2}(t-t')| > |t-t'|) \end{cases} \tag{9.37}$$

for constants c_3, c_4. Then $\|k_2(t,t',\cdot)\|_{a,b} \leq c_5|t-t'|^{1/a}$. The conditions of Theorem 9.2(c) are satisfied with $\eta = 1/a > 1/\alpha(u)$, so strong localisability follows. ■

Note that the differentiability conditions in Theorem 9.3 could be weakened slightly to Hölder conditions for which Theorem 9.2 would still be applicable.

Recall that an α-stable Lévy motion, $0 < \alpha < 2$, is a process of $D(\mathbb{R})$ with stationary independent increments which have a strictly α-stable distribution. Taking M as a symmetric α-stable random measure on $\mathbb{R}$, the α-stable Lévy motion may be represented as

$$L_\alpha(t) = M[0,t] = \int \mathbf{1}_{[0,t]}(x) M(dx) = c(\alpha) \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} \mathbf{1}_{[0,t]}(\mathsf{X})\mathsf{Y}^{<-1/\alpha>} \quad (t \in \mathbb{R}), \tag{9.38}$$

where Π is the Poisson process on $\mathbb{R}^2$ with $\mathcal{L}^2$ as mean measure, $\mathbf{1}_{[0,t]}$ is the indicator function and $c(\alpha) = \big(2\alpha^{-1}\Gamma(1-\alpha)\cos(\frac{1}{2}\pi\alpha)\big)^{-1/\alpha}$. Then L_α is $1/\alpha$-sssi and is strongly $1/\alpha$-localisable in $D(\mathbb{R})$.

Theorem 9.4 *(Multistable Lévy motion). Let $\alpha : \mathbb{R} \to (0,2)$ and $a : \mathbb{R} \to \mathbb{R}^+$ be continuously differentiable, and define*

$$Y(t) = a(t)c(\alpha(t)) \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} \mathbf{1}_{[0,t]}(\mathsf{X})\mathsf{Y}^{<-1/\alpha(t)>} \quad (t \in \mathbb{R}). \tag{9.39}$$

(a) If $1 < \alpha(u) < 2$ then Y is $1/\alpha(u)$-localisable at u, with $Y'_u = a(u)L_{\alpha(u)}$.

(b) If $0 < \alpha(u) < 1$ and $\alpha'(u) \neq 0$ then Y is 1-localisable at u with $\{Y'_u(t) : t \in \mathbb{R}\} = \{tW : t \in \mathbb{R}\}$, where W is the random variable

$$W = a(u)c(\alpha(u)) \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} \mathbf{1}_{[0,u]}(\mathsf{X})\mathsf{Y}^{<-1/\alpha(u)>} \left(\frac{\alpha'(u)}{\alpha(u)^2}|\log|\mathsf{Y}|| + \frac{d}{du}(a(u)c(\alpha(u)))\right).$$

Proof. (a) By Proposition 3.2 the term $a(t)c(\alpha(t))$ does not affect localisability. Define a random field by

$$X(t,v) = \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} \mathbf{1}_{[0,t]}(\mathsf{X})\mathsf{Y}^{<-1/\alpha(v)>} \quad (t, v \in \mathbb{R}).$$

Taking $f(t,v,x) = \mathbf{1}_{[0,t]}(x)$ the conditions of Theorem 9.2(a) are satisfied with $h = 1/\alpha(u) < 1 \equiv \eta$, so the result follows from the localisability of L_α.

(b) In the case where $a(t)c(\alpha(t)) = 1$

$$\frac{Y(u+rt)-Y(u)}{r} = \frac{1}{r}\sum_{(\mathsf{X},\mathsf{Y})\in\Pi} \mathbf{1}_{[0,u]}(\mathsf{X})(\mathsf{Y}^{<-1/\alpha(u+rt)>} - \mathsf{Y}^{<-1/\alpha(u)>})$$
$$+\frac{1}{r}\sum_{(\mathsf{X},\mathsf{Y})\in\Pi} \mathbf{1}_{[u,u+rt]}(\mathsf{X})\mathsf{Y}^{<-1/\alpha(u+rt)>}.$$

Letting $r \to 0$ the second term vanishes if $1/\alpha(u) > 1$ and the first term converges to W in finite dimensional distributions. The general case is similar. ∎

Note that Theorem 9.4(b) illustrates a general phenomenon that occurs when the process $\{X(t,u) : t \in \mathbb{R}\}$ is $h(u)$-localisable at u where $h(u) > 1$. The process $\{Y'_u(t) : t \in \mathbb{R}\}$ will typically be 1-localisable at u, with the dominant component of $Y'_u(t)$ derived from $(X(u+rt, u+rt) - X(u+rt, u))/r$ rather than from $X'_u(t,u)$.

As explained in [17, Section 7.6], there are two ways to extend the linear fractional stable motion to the case $H = 1/\alpha$. Apart from the Lévy motion considered above, one may define the following process, called log-fractional stable motion:

$$\Lambda_\alpha(t) = \int_{-\infty}^{\infty} (\log(|t-x|) - \log(|x|))\, M(dx) \quad (t \in \mathbb{R}) \tag{9.40}$$

where, as usual, M is an α-stable random measure. This process is well-defined only for $\alpha \in (1,2]$ (the integrand does not belong to $\mathcal{F}_\alpha$ for $\alpha \le 1$). It is $1/\alpha$-self-similar with stationary increments. Unlike the Lévy motion, however, its increments are not independent. Another difference is that log-fractional stable motion does not have a version in $D(\mathbb{R})$, so we cannot speak of strong localisability.

Theorem 9.5 *(Log-fractional multistable motion) Let $\alpha : \mathbb{R} \to (1,2)$ and a be continuously differentiable, and define*

$$Y(t) = a(t)\sum_{(\mathsf{X},\mathsf{Y})\in\Pi} (\log|t-\mathsf{X}| - \log|\mathsf{X}|)\,\mathsf{Y}^{<-1/\alpha(t)>} \quad (t \in \mathbb{R}). \tag{9.41}$$

Then Y is $1/\alpha(u)$-localisable at all $u \in \mathbb{R}$, with $Y'_u = a(u)\Lambda_{\alpha(u)}$.

Proof. The proof is similar to that of Theorem 9.4, by considering the field

$$X(t,v) = \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} (\log|t-\mathsf{X}| - \log|\mathsf{X}|)\,\mathsf{Y}^{<-1/\alpha(v)>} \quad (t, v \in \mathbb{R}), \tag{9.42}$$

with Theorem 9.2(a) is applied to $f(t,v,x) = \log|t-x| - \log|x|$. ∎

For a final example we give a multistable version of Theorem 7.6

Theorem 9.6 *(Multistable reverse Ornstein-Uhlenbeeck process) Let $\lambda > 0$ and $\alpha : \mathbb{R} \to (1,2)$ be continuously differentiable. Let*

$$Y(t) = \sum_{(\mathsf{X},\mathsf{Y})\in\Pi,\mathsf{X}\ge t} \exp(-\lambda(\mathsf{X}-t))\mathsf{Y}^{<-1/\alpha(t)>} \quad (t \in \mathbb{R}).$$

Then Y is $1/\alpha(u)$-localisable at all $u \in \mathbb{R}$, with $Y'_u = c(\alpha(u))^{-1}L_{\alpha(u)}$, where L_α is Lévy α-stable motion.

Proof. Let

$$X(t,v) = \sum_{(\mathsf{X},\mathsf{Y})\in\Pi, \mathsf{X}\geq t} \exp(-\lambda(\mathsf{X}-t))\mathsf{Y}^{<-1/\alpha(v)>} \quad (t,v\in\mathbb{R}). \tag{9.43}$$

Then for each v, $X(\cdot,v)$ is everywhere $1/\alpha(v)$-localisable with $X'_u(\cdot,v) = c(\alpha(v))^{-1}L_{\alpha(v)}(\cdot)$ by Theorem 7.6. Applying Theorem 9.2(a) with $f(t,v,x) = \mathbf{1}_{[t,\infty)}\exp(-\lambda(x-t))$ gives the conclusion. ■

10 Further work

There are a great many possible variants and extensions of this work. Localisable processes of many other forms may be constructed. For example multistable processes with skewness and the class of stationary localisble processes deserve investigation. There may be advantages in seeking other representations of multistable processes such as by sums involving arrival times of a Poisson process or as stochastic integrals with respect to suitably constructed random measures. Our conditions for localisability could certainly be weakened and further techniques for establishing localisability and in particular strong localisability developed. Effective techniques for simulation and inference on parameters for these processes are also needed. We will be addressing some of these matters in a sequel to this paper.

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